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Magneto-Fluid Dynamics Division

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Nearly Axisymmetric Magnetic Field

Harold Weitzner

Courant Institute of Mathematical Sciences

New York University

251 Mercer Street

New York, N. Y. 10012

ABSTRACT

The motion of a charged particle in a static magnetic field is studied by means of repeated canonical transformations of a Hamiltonian system. Adiabatic invariants are generated based on the assumption that the particle larmor radius is small compared with the characteristic distance over which the magnetic field varies. Unlike many earlier treatments the transformations presented here preserve the axisymmetry of the dynamics when the magnetic field is axisymmetric. It is assumed that the magnetic field consists of a small nonaxisymmetric part plus the axisymmetric toroidal and poloidal parts. After the introduction of the magnetic moment adiabatic invariant the motion of the guiding center is studied. The results depend sensitively on the ratio of the poloidal magnetic field to the total magnetic field. In some cases a second adiabatic invariant exists and direct inferences concerning long time particle drifts are possible. In one case where a second adiabatic invariant fails to exist, long term drifts are studied by conventional perturbation expansions. At some points resonance or lack of resonance phenomena appear and determine the drift effects.

I. Introduction

The motion of a single charged particle in a given magnetic field, one of the first problems studied in plasma physics,⁽¹⁾ has recently become of interest again. Two topics of immediate significance in magnetic fusion energy research have sparked the current concern:⁽²⁾ the confinement of high energy alpha particles and the drift motion of moderate energy particles induced by small magnetic fields that break axial symmetry. The large body of older theory has been used to provide a base for extensive numerical computations. Here, we examine the problem with analytic tools to discover what improved descriptions of single particle motion we are able to provide. The critical concept in the study of the single particle dynamics is the existence of adiabatic invariants of the motion which allows great simplifications in the study of the dynamics. The adiabatic invariants are associated with the presence of a physical parameter which is small, the ratio of the larmor radius to the characteristic distance over which magnetic fields vary; and we denote this parameter by ϵ . Unlike the earlier treatments we assume that parts of the axisymmetric magnetic field may be small in ϵ and we also take the symmetry breaking field to be small in ϵ . Not surprisingly, we find that these assumptions do allow considerably more analysis than is possible without them.

The older work has shown that provided ϵ is small and both B and $1/B$ are of order 1 in ϵ , where B is the given magnetic field, then there is a first adiabatic invariant, the magnetic moment of the particle. Using the approximate constancy of the magnetic moment, one may eliminate part of the single particle motion and study the simpler motion of the guiding centers of the particles. In many cases a second

adiabatic invariant exists as well, and this second invariant, often called the longitudinal invariant, may be used to characterize the slow drifts of the guiding centers across the magnetic field lines. We are not concerned with the third or flux invariant, which also may exist in some cases.

The motion of a single particle is given by Hamilton's equations with the standard Hamiltonian for a single charged particle in a given magnetic field. C. S. Gardner⁽³⁾ has applied the particularly powerful and elegant procedure of repeated canonical transformations to this problem. This technique gives a systematic perturbation procedure which preserves the Hamiltonian character of the system and allows the elimination of degrees of freedom as the adiabatic invariants are introduced. The methods allow the isolation of drift motions and give reliable estimates of time intervals over which results apply. A major difficulty with the use of canonical transformations is that the great freedom of choice of transformations leaves the user unsure of how to start. After the first step, the process is fairly systematic, but that first step is far from obvious or natural. As first carried out by Gardner, the procedure had the unfortunate property that the axisymmetry of particle motion in axisymmetric magnetic fields was destroyed. This difficulty has occurred in other treatments as well,⁽⁴⁾ although there is an ad hoc device to eliminate the problem in lowest order.⁽⁵⁾ One of our purposes is to present consistent perturbation schemes which preserve axisymmetry when present but allow for its absence. Our other intention is to examine drift motion in three distinct cases. We find the use of repeated canonical

transformations ideally suited to our goals. Our treatment is based on Gardner's approach.

In order to discuss the drift motion problem we must describe the magnetic field more carefully. We treat the case of static magnetic fields with no electric field; time dependent magnetic fields or electric fields are readily introduced following Gardner's paradigm. We introduce cylindrical coordinates (r, θ, z) and we assume that the magnetic field $\mathbf{B}$ is the sum of an axisymmetric magnetic field independent of θ and a nonaxisymmetric field $\mathbf{B}_{NA}$. The axisymmetric field is itself the sum of a toroidal field $\hat{\theta} B_T$ plus the poloidal field $\mathbf{B}_p$ with components in the $\hat{r}$ and $\hat{z}$ directions only. The flux surfaces we mention subsequently are those of the axisymmetric field. We assume throughout that $B_{NA}/B \sim \epsilon$. We find that the drift motion depends critically on the ratio B_p/B . We consider separately the three cases $B_p/B \sim 1$, $B_p/B \sim \sqrt{\epsilon}$, and $B_p/B \sim \epsilon$. In each case we obtain canonically conjugate coordinates and momenta which show that the motion is compounded of several motions on differing time scales. The gyro rotation of the particle about its guiding center is on the fastest time scale in each case. The introduction of the magnetic moment adiabatic invariant effectively eliminates this motion from the Hamiltonian and drops the number of pairs of canonically conjugate variables from three to two. The reduced Hamiltonian describes the motion of the guiding center. We now consider the dynamics of the guiding center.

When $B_p/B \sim 1$ there are two classes of guiding center orbits. The passing particles, or untrapped particles, encircle the magnetic axis while the trapped particles do not encircle the magnetic axis. The

trapped particles possess a second adiabatic invariant, from whose existence we show that the guiding centers are bounded to lie near a given flux surface for very long times. Only nonadiabatic effects will allow significant drifts across flux surfaces. The description of the passing particles is delicate and our analysis does not give a full picture. For certain very special classes of magnetic fields a second adiabatic invariant exists and from its existence one may construct a Hamiltonian for the very slow particle drifts across flux surfaces. These drifts may indeed be large. The condition for these results is a particular approximate resonance in the motion. If the resonance is not present then no second adiabatic invariant exists and we are unable to describe the drifts easily. It becomes clear that in such rather general cases high order accuracy of the guiding center motion is needed to obtain the drifts correctly. This scaling is discussed in the third section.

When $B_p/B \sim \epsilon$ there is always a second adiabatic invariant. The dependence of the second adiabatic invariant on the canonical angular momentum p_θ and the relation between p_θ and the flux surface on which a particle lies are drastically different from the case just discussed. The drift orbits, even without nonaxisymmetric perturbing fields, are large and show large deviations from a given flux surface. The only significant effect of the nonaxisymmetric fields is to rearrange the orbits near the critical points in the phase plane of the drift Hamiltonian. This scaling is treated in the fourth section.

The intermediate scaling $B_p/B \sim \sqrt{\epsilon}$ is the most difficult of the three. With one set of canonical variables it seems that a second adiabatic invariant exists. However, it is shown that the time

interval over which its constancy is valid is severely limited. While it might appear on the limited time interval that the invariant is constant, its value must gradually change. With another set of canonically conjugate variables no second adiabatic invariant can be constructed. However, the motion can be described further by means of a more or less conventional perturbation expansion. Again the orbits separate into trapped and untrapped particles. The trapped particle drift across the flux surface per particle period is evaluated. Thus, the drift may be described as a discrete process. Summed over many periods resonance coherence effects may enhance the drift, or lack of resonance may average the drifts to an effect of much smaller order of magnitude. We have estimated the smaller nonresonant drifts and append a few comments of a purely qualitative nature. The untrapped particle drifts are only briefly alluded to in order to indicate that small, nonresonant drifts will be present here too. The analysis of this scaling is presented in section five. In addition, an appendix contains many of the details of the canonical transformations to high enough order to obtain the guiding center Hamiltonian.

The essential element that allows the preservation of axisymmetry when appropriate is the introduction of particular parametrizations of the vector potential of the magnetic field. The usual treatments employ the so called (α, β, s) Clebsch variables, $\mathbf{B} = \nabla\alpha \times \nabla\beta$. In the second section we present the gauge transformation of the vector potential which generates the several coordinate functions we use.

It may be of use to point out to the reader the origins of the complexity of the particle dynamics we have attempted to describe. There are two major, and distinct problems. The first problem relates

to the question of when an adiabatic invariant exists. In order to have an adiabatic invariant one must have two distinct properties of the Hamiltonian. First, one must have the motion a compound of motions on different time scales, and second, one must have a family of approximating, periodic orbits on the fastest time scale. In some cases the family of approximating, periodic orbits may not exist and then no invariant exists. The other problem relates to the parametrization of the magnetic field and the selection of canonically conjugate variables. When one scales components of the magnetic field in the small parameter ϵ , the order of magnitude of canonical variables shifts too, and the order of magnitude of canonical variables affects the time scales on which variables change. Further, variables that are originally functionally independent may become, to leading order, functionally dependent. The implications of these effects could be seen in the older works; they just had not been examined with any thoroughness.

II. Kinematic Preliminaries

In this section we introduce the appropriate gauge for the vector potential associated with a given magnetic field and the small larmor radius scalings for the various magnetic field scalings. We employ cylindrical coordinates r, θ, z throughout. Let $\tilde{\mathbf{A}}$ be a vector potential for a given magnetic field $\mathbf{B}$,

$$\mathbf{B} = \nabla \times \tilde{\mathbf{A}} .$$

We might equally well set

$$\mathbf{B} = \nabla \times \mathbf{A}' \quad (1)$$

where $\mathbf{A}' = \tilde{\mathbf{A}} + \nabla \beta(r, \theta, z) + \nabla \gamma(r, z)$. We introduce the notation

$$\langle f(r, \theta, z) \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta', z) d\theta' , \quad (2)$$

and clearly $\langle f(r, \theta, z) \rangle$ is independent of θ . We see that

$$\begin{aligned} A'_\theta(r, \theta, z) &= \langle \tilde{A}_\theta(r, \theta, z) \rangle + (\tilde{A}_\theta(r, \theta, z) - \langle \tilde{A}_\theta(r, \theta, z) \rangle) \\ &\quad + \beta_{,\theta}(r, \theta, z)/r , \end{aligned}$$

so that if we set

$$\beta(r, \theta, z) = r \int_0^\theta d\theta' (\langle \tilde{A}_\theta(r, \theta, z) \rangle - \tilde{A}_\theta(r, \theta', z)) ,$$

where $\beta(r, \theta, z)$ is a periodic function of theta, and

$$\psi(r, z) = \langle \tilde{A}_\theta(r, \theta, z) \rangle / r ,$$

then

$$A'_0(r, \theta, z) = \psi(r, z)/r. \quad (3)$$

We now turn to the specification of $\gamma(r, z)$. We seek to find $\gamma(r, z)$ and $\lambda(r, z)$ such that

$$\begin{aligned} \langle \tilde{A}_r \rangle + \langle \beta \rangle_{,r} + \gamma_{,r}(r, z) &= \lambda(r, z) \psi_{,r}(r, z) \\ \langle \tilde{A}_z \rangle + \langle \beta \rangle_{,z} + \gamma_{,z}(r, z) &= \lambda(r, z) \psi_{,z}(r, z) . \end{aligned}$$

A necessary and sufficient condition that such a $\gamma(r, z)$ exist is that

$$\langle \tilde{A}_r \rangle_{,z} - \langle \tilde{A}_z \rangle_{,r} = \gamma_{,z} \psi_{,r} - \lambda_{,r} \psi_{,z} ,$$

or

$$\langle B \rangle \cdot \nabla \lambda(r, z) = r \langle B_\theta(r, \theta, z) \rangle. \quad (4)$$

We may use (4) to determine $\lambda(r, z)$ although $\lambda(r, z)$ will not, in general, be single-valued. With $\lambda(r, z)$ satisfying (4) and $\gamma(r, z)$ chosen consistently we have obtained the following representation of the vector potential

$$A' = \lambda(r, z) \nabla \psi(r, z) + \hat{\theta} \psi(r, z)/r + \hat{r} A_r(r, \theta, z) + \hat{z} A_z(r, \theta, z) \quad (5)$$

where

$$\langle A_r(r, \theta, z) \rangle = \langle A_z(r, \theta, z) \rangle = 0. \quad (6)$$

We identify the magnetic field generated by $\lambda(r, z)$ and by $\psi(r, z)$ as the axisymmetric magnetic field, while $A_r(r, \theta, z)$ and $A_z(r, \theta, z)$

generate the nonaxisymmetric field. Thus, we find that the axisymmetric toroidal field is

$$B_T = \hat{\theta} (\lambda_{,z}\psi_{,r} - \lambda_{,r}\psi_{,z}) ,$$

while the axisymmetric poloidal field is

$$B_P = -\hat{r}\psi_{,z}/r + \hat{z}\psi_{,r}/r ,$$

so that we may interpret $\psi(r,z)$ as the usual flux function. Finally, the nonaxisymmetric field has the representation

$$B_{NA} = \hat{r} A_{z,\theta}/r + \hat{\theta} (A_{r,z} - A_{z,r}) - \hat{z} A_{r,\theta}/r .$$

We shall make use of one further function, $S(r,z)$, introduced so that the curves $S(r,z) = \text{const.}$ are orthogonal to the curves $\psi(r,z) = \text{const.}$

$$0 = \nabla\psi \cdot \nabla S = \psi_{,r}S_{,r} + \psi_{,z}S_{,z} . \quad (7)$$

We next introduce the small larmor radius scaling. We adopt the convention that B is $O(1)$ in the larmor radius parameter, and more precisely that both B and $1/B$ are $O(1)$; that is B may not become small or large. Since the small parameter ϵ is the ratio of the larmor radius to characteristic distance of magnetic field variation, we may take either the larmor radius as small, or the characteristic distance of magnetic field variation as large. It is, in fact, simpler to take the larmor radius itself as $O(1)$ and the other parameter as large. Thus, if we have a function of space $f(x,y,z,\epsilon)$, then if the distance of spatial variation were large, we would expect f to have the representation $f(x,y,z,\epsilon) = F(\epsilon x, \epsilon y, \epsilon z, \epsilon)$ where $F(X,Y,Z,E)$ has all its derivatives of order 1 with respect to its arguments X,Y,Z,E . Under these assumptions $\nabla f = \epsilon (F_{,\epsilon X} \hat{x} + F_{,\epsilon Y} \hat{y} + F_{,\epsilon Z} \hat{z})$ and we see that f

changes slowly in x, y, z . The analogous representation for a function in polar coordinates $g(r, \theta, z, \epsilon)$ would be

$$g(r, \theta, z, \epsilon) = G(\epsilon r, \theta, \epsilon z, \epsilon)$$

where again $G(R, \theta, Z, E)$ would have all derivatives of order 1 with respect to the arguments. Now we find

$$\nabla g = \epsilon (G_{,\epsilon r} \hat{r} + G_{,\theta} \hat{\theta} / (\epsilon r) + G_{,\epsilon z} \hat{z})$$

so that we require, in general that $1/(\epsilon r)$ be $O(1)$ in order that ∇g be $O(\epsilon)$. That is, we must assume that we are far enough from the axis of the cylindrical coordinate system that ϵr not be small. We know that near the axis quantities may vary rapidly so this restriction is not surprising or unreasonable. Physically, we must be many larmor radii from the axis.

We now give the scaled versions of the vector potential for the several cases. When $B_T \sim 1$ or smaller, $B_p \sim 1$ and $B_{NA} \sim \epsilon$ we set

$$\begin{aligned} \vec{A}' = & \lambda(\epsilon r, \epsilon z) (\hat{r} \psi_{,\epsilon r}(\epsilon r, \epsilon z) + \hat{z} \psi_{,\epsilon z}(\epsilon r, \epsilon z)) / \epsilon \\ & + \hat{\theta} \psi(\epsilon r, \epsilon z) / (\epsilon^2 r) + A_r(\epsilon r, \theta, \epsilon z) \hat{r} + A_z(\epsilon r, \theta, \epsilon z) \hat{z} \end{aligned} \quad (8)$$

where again $\langle A_r(\epsilon r, \theta, \epsilon z) \rangle = \langle A_z(\epsilon r, \theta, \epsilon z) \rangle = 0$. We find

$$B_T = \hat{\theta} (\lambda_{,\epsilon z} \psi_{,\epsilon r} - \lambda_{,\epsilon r} \psi_{,\epsilon z}) \quad (9)$$

$$B_p = -\hat{r} \psi_{,\epsilon z} / (\epsilon r) + \hat{z} \psi_{,\epsilon r} / (\epsilon r), \quad (10)$$

and

$$B_{NA} = \epsilon \{ \hat{r} A_{z,\theta} / (\epsilon r) + \hat{\theta} (A_{r,\epsilon z} - A_{z,\epsilon r}) - \hat{z} A_{r,\theta} / (\epsilon r) \} \quad (11)$$

so that provided (ϵr) is $O(1)$ and not smaller $B_T = O(1)$, $B_p = O(1)$ and $B_{NA} = O(\epsilon)$. Thus, the factors of ϵ in the denominators of (8) are

needed to make B , which is a derivative of A' , $O(1)$ and not smaller. If we multiply the θ component of (8) by ϵ^q then the poloidal component of B is multiplied by ϵ^q . Thus, if we wish $B_p \sim \epsilon$ ($B_p \sim \sqrt{\epsilon}$) then we multiply the θ component of (8) by ϵ ($\sqrt{\epsilon}$).

The Hamiltonian for the single particle motion is

$$H = (1/2)\{(p_r - A'_r)^2 + (p_\theta/r - A'_\theta)^2 + (p_z - A'_z)^2\} \quad (12)$$

and

$$\begin{aligned} v_r &= p_r - A'_r \\ v_\theta &= p_\theta/r - A'_\theta \\ v_z &= p_z - A'_z \end{aligned} \quad (13)$$

Since we have taken $B \sim 1$ and the larmor radius $O(1)$ we conclude that $\underline{y} \sim 1$ as well. We are now ready to start the study of particle dynamics.

III. The Guiding Center Expansion in the General Case

In the general magnetic field configuration we take $B_p \sim 1$, $B_T \lesssim 1$ while $B_{NA} \sim \epsilon$. Our results correspond to the generally expected picture of single particle motion. After an appropriate symmetry preserving canonical transformation we show that the dynamics is composed of motions on three distinct time scales. The fastest motion is the gyro rotation of the particle about its guiding center. The introduction of the magnetic moment adiabatic invariant by successive canonical transformations eliminates two degrees of freedom and the fastest time scale from the Hamiltonian motion. The next slower motion is the guiding center motion. On this time scale there are two classes of orbits, trapped (or banana) orbits for which the trajectory is essentially bounded and periodic in the appropriate coordinate, and untrapped orbits which may or may not be periodic in the appropriate coordinates. In either case the motion on this time scale is essentially on a given flux surface. In the case of axisymmetry the slowest time scale is largely irrelevant to the motion and the preceding description is complete with particles effectively constrained to be near a given flux surface. Small, nonaxisymmetric perturbations raise the possibility of significant particle drifts on the slowest time scale. We show that for the trapped particles a second adiabatic invariant exists and that its existence allows the elimination of two more degrees of freedom from the Hamiltonian. Nevertheless, we show that on the slowest time scale particles are effectively confined to a given flux surface. For the untrapped particles a second invariant may or may not exist depending on the existence or nonexistence of resonances between two modes of motion.

The effects of resonances are explored, albeit not conclusively. We see that it is possible for an adiabatic invariant to exist leading to a Hamiltonian on the slowest time scale with large particle drifts. Equally well the drifts could be small, or even no second adiabatic invariant might exist. The origin of the complexity is quite simple. In order that an adiabatic invariant exist it is necessary not only that there be a separation of time scales but also that on the faster scale the motion must be periodic. We have the separation of time scales but we may not have periodic orbits.

We may readily construct the Hamiltonian for the motion from (8) and (12) as

$$H = (1/2) \{ (p_r - \lambda \psi_{, \epsilon r} / \epsilon - A_r)^2 - (p_z - \lambda \psi_{, \epsilon z} / \epsilon - A_z)^2 + (\epsilon p_\theta - \psi / \epsilon)^2 / (\epsilon r)^2 \}. \quad (14)$$

We consider ϵr and ϵz as functions of ψ and S so that $\lambda(\epsilon r, \epsilon z)$, $\psi_{, \epsilon r}(\epsilon r, \epsilon z)$ and $\psi_{, \epsilon z}(\epsilon r, \epsilon z)$ are functions of ψ and S or $\lambda = \lambda(\psi, S)$, etc. We introduce the generator of a transformation $F(p_1, p_2, p_3, r, \theta, z)$ involving the old coordinates r, θ, z and new momenta p_1, p_2, p_3

$$F = (\psi / \epsilon - \epsilon p_1)(p_3 + \lambda / \epsilon) + S p_2 / \epsilon + \theta p_1 - \lambda_{, \psi} (\psi / \epsilon - \epsilon p_1)^2 / 2 \quad (15)$$

and from the relations

$$q_i = \frac{\partial F}{\partial p_i}, \quad i = 1, 2, 3$$

$$(p_r, p_\theta, p_z) = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right) F$$

we readily conclude

$$\begin{aligned}
 \psi &= \varepsilon^2 p_1 + \varepsilon q_3 \\
 S &= \varepsilon q_2 \\
 \theta &= q_1 + \lambda(\psi, S) + \varepsilon p_3 - \varepsilon q_3 \lambda_{,\psi}(\psi, S)
 \end{aligned}
 \tag{16}$$

and

$$\begin{aligned}
 p_\theta &= p_1 \\
 p_r - \lambda_{\psi, \varepsilon r} / \varepsilon &= \psi_{, \varepsilon r} (p_3 - \varepsilon q_3^2 \lambda_{, \psi \psi} / 2) \\
 &\quad + S_{, \varepsilon r} (p_2 + q_3 \lambda_{, S} - \varepsilon q_3^2 \lambda_{, \psi S} / 2) \\
 p_z - \lambda_{\psi, \varepsilon z} / \varepsilon &= \psi_{, \varepsilon z} (p_3 - \varepsilon q_3^2 \lambda_{, \psi \psi} / 2) \\
 &\quad + S_{, \varepsilon z} (p_2 + q_3 \lambda_{, S} - \varepsilon q_3^2 \lambda_{, \psi S} / 2) .
 \end{aligned}
 \tag{17}$$

Following the ideas of Gardner we must now determine the magnitude (in ε) of the new canonical variables (p_i, q_i) , $i = 1, 2, 3$. That is, in the time evolution of the particle trajectories, the variables cover some range of values, and we must estimate that range. We recall that we have scaled the motion so that $V \sim 1$, in particular $V_\theta \sim 1$. We readily find from (8), (13), (16) and (17) that

$$V_\theta = -q_3 / (\varepsilon r) .$$

Since we assume that we are far from the axis $r = 0$, $\varepsilon r \sim 1$ and $1/(\varepsilon r) \sim 1$, so that we conclude $q_3 = O(1)$. The analogous argument applied to V_r and V_z leads to the conclusion that p_2 and p_3 also are $O(1)$ in ε . Now both S and ψ are coordinates of points in physical space and we expect them both to have the range $O(1)$ in ε . Thus, we see from (16) that $\varepsilon^2 p_1$ and εq_2 are $O(1)$. The range of θ and q_1 are essentially the

same and may be large in ϵ if the particle makes a large number of transits around the torus. Since the Hamiltonian is periodic in θ , and hence q_1 , large values of either variable leave the Hamiltonian and all its derivatives $O(1)$. We treat $q_1 = O(1)$ although it may become large. In summary, the combinations $\epsilon^2 p_1$, q_1 , p_2 , ϵq_2 , p_3 and q_3 are all $O(1)$. These properties are essential to the arguments that follow.

It will become clear shortly that (p_3, q_3) are essentially the component of $v_\perp$, the gyro rotation velocity, p_2 is related to $v_\parallel$ while $\epsilon q_2 = S$ is the position of a particle on a magnetic field line, and (p_1, q_1) are the canonical angular momentum and angular position, respectively. We may also interpret $\epsilon^2 p_1$ as essentially the flux coordinate of the particle. We may insert (16) and (17) into (14) to find the new Hamiltonian as

$$H = (1/2) \{ E(\psi, S) (p_3 - \epsilon q_3^2 \lambda_{,\psi\psi}/2 - A_\psi(\psi, S, \theta))^2 + G(\psi, S) (p_2 + q_3 \lambda_{,S} - \epsilon q_3^2 \lambda_{,\psi S}/2 - A_S(\psi, S, \theta))^2 + R(\psi, S) q_3^2 \} \quad (18)$$

where

$$\begin{aligned} E &= \psi_{,\epsilon r}^2 + \psi_{,\epsilon z}^2 \\ G &= S_{,\epsilon r}^2 + S_{,\epsilon z}^2 \\ R &= 1/(\epsilon r)^2 \end{aligned} \quad (19)$$

and

$$\hat{r}A_r + \hat{z}A_z = A_\psi(\nabla\psi/\epsilon) + A_S(\nabla S/\epsilon) \quad (20)$$

E , G , R are first evaluated as functions of ϵr , ϵz and then ϵr , ϵz are considered as functions ψ and S . Since $\nabla\psi/\epsilon$ and $\nabla S/\epsilon$ are both $O(1)$, A_ψ and A_S are $O(1)$ and they are treated similarly first as functions of

ϵr , ϵz , θ and then as functions of ψ , S , and θ . Finally in (18) we employ (16) to replace ψ , S , and θ in H by the new canonical variables (p_i, q_i) , $i = 1, 2, 3$. If $A_\psi = A_S = 0$ so that no nonaxisymmetric fields appear, the Hamiltonian is independent of q_1 so that $p_1 \equiv \text{const.}$ and axisymmetry is preserved. In the usual treatment the Clebsch variables α , β , s are used as coordinates, $\mathbf{B} = \nabla\alpha \times \nabla\beta$ and the coordinates θ enters in a nonignorable manner in $\alpha(r, \theta, z)$ or $\beta(r, \theta, z)$.

We see that the essential structure of our Hamiltonian is

$$H = H(\epsilon^2 p_1, q_1, p_2, \epsilon q_2, p_3, q_3, \epsilon)$$

where derivatives of H with respect to its arguments are all $O(1)$. We may rewrite Hamilton's equations as

$$(\epsilon^2 \dot{p}_1) = -\epsilon^2 \left(\frac{\partial H}{\partial q_1} \right)$$

$$\dot{q}_1 = \epsilon^2 \left(\frac{\partial H}{\partial \epsilon^2 p_1} \right)$$

$$\dot{p}_2 = -\epsilon \left(\frac{\partial H}{\partial \epsilon q_2} \right)$$

$$(\epsilon \dot{q}_2) = \epsilon \frac{\partial H}{\partial p_2}$$

$$\dot{p}_3 = -\frac{\partial H}{\partial q_3}$$

$$\dot{q}_3 = \frac{\partial H}{\partial p_3}.$$

Hence, over a time interval $O(1)$ Δp_3 and $\Delta q_3 \sim 1$ while Δp_2 and $\Delta(\epsilon q_2) \sim \epsilon$, but $\Delta \epsilon^2 p_1$ and $\Delta q_1 \sim \epsilon^2$. Provided p_3 and q_3 remain $O(1)$ over the longer time interval $O(1/\epsilon)$, Δp_2 and $\Delta(\epsilon q_2) \sim 1$. Finally over the longest time interval $O(1/\epsilon^2)$, $\Delta \epsilon^2 p_1$ and Δq_1 are $O(1)$. Thus, our variables (p_i, q_i) have decomposed the motion into motion on the three distinct time scales 1 , $1/\epsilon$, and $1/\epsilon^2$. This decomposition holds even in the axisymmetric case $A_S = A_\psi = 0$.

We are now prepared to construct the various adiabatic invariants

following Gardner's procedure. If we hold $\epsilon^2 p_1$, q_1 , p_2 and ϵq_2 fixed we see that the curves $H = \text{const.}$ are ellipses in the p_3 , q_3 plane. We may distort the ellipses into circles by an affine transformation in the p_3 , q_3 plane and we may generate such an affine transformation by a canonical transformation in the p_3 , q_3 plane. We may also, then, extend the transformation into the full (p_i, q_i) , $i = 1, 2, 3$ space. Specifically, we introduce the generator of the transformation from (p_i, q_i) , $i = 1, 2, 3$ to $(\bar{p}_i, \bar{q}_i)$, $i = 1, 2, 3$,

$$F = (\alpha \bar{p}_3 + A_\psi)(q_3 + \gamma) + \bar{p}_1 q_1 + \bar{p}_2 q_2 \quad (21)$$

where

$$E\alpha^2 = (B_T^2 + B_P^2)^{1/2} \quad (22)$$

$$\gamma = G(\bar{p}_2 - A_S)\lambda_{,S}/(G\lambda_{,S}^2 + R)$$

and the several functions α , γ , E , G , R , B_P , B_T , $\lambda_{,S}$, A_ψ and A_S are first evaluated as spatial functions of ψ , S , θ followed by the replacements

$$\psi = \epsilon^2 \bar{p}_1 + \epsilon q_3$$

$$S = \epsilon q_2$$

$$\theta = q_1 + \lambda(\psi, S) + \epsilon \bar{p}_3 - \epsilon q_3 \lambda_{,\psi}.$$

We find easily

$$\epsilon^2 p_1 = \epsilon^2 \bar{p}_1 + O(\epsilon^2)$$

$$q_1 = \bar{q}_1 + O(\epsilon^2)$$

$$p_2 = \bar{p}_2 + O(\epsilon)$$

$$\begin{aligned}\epsilon q_2 &= \epsilon \bar{q}_2 + O(\epsilon) \\ p_3 &= \alpha \bar{p}_3 + A_\psi + O(\epsilon) \\ q_3 &= \bar{q}_3/\alpha - \gamma + O(\epsilon)\end{aligned}$$

so that the barred and original variables are of the same order in ϵ . We could easily but tediously evaluate the error terms to any desired order in ϵ . The transformed Hamiltonian becomes

$$H = (1/2)(\bar{p}_3^2 + \bar{q}_3^2)B + (1/2)G(\bar{p}_2 - A_S)^2 B_P^2/B^2 + O(\epsilon) \quad (23)$$

where

$$B = (B_P^2 + B_T^2)^{1/2} \quad (24)$$

and B_P , B_T , B , G and A_S are first evaluated as functions of ψ , λ , θ followed by the replacements $\psi = \epsilon^2 \bar{p}_1$, $S = \epsilon \bar{q}_2$, $\theta = \bar{q}_1 + \lambda(\psi, S)$.

We may now readily repeat the process. The terms proportional to ϵ in the Hamiltonian (23) are a third degree polynomial in $(\bar{p}_3, \bar{q}_3)$. With an appropriate canonical transformation (see the appendix where an analogous transformation is carried out in some detail) we find

$$\begin{aligned}H &= (1/2)(B + \epsilon L)(\bar{\bar{p}}_3^2 + \bar{\bar{q}}_3^2) + (1/2)G(\bar{\bar{p}}_2 - A_S)^2 B_P^2/B^2 \\ &\quad + \epsilon M(\bar{\bar{p}}_2, \epsilon \bar{\bar{q}}_2, \epsilon^2 \bar{\bar{p}}_1, \bar{\bar{q}}_1) + O(\epsilon^2)\end{aligned}$$

where B_P , B_T , B , L , G , and A_S are evaluated at ψ , S , θ followed by the replacements $\psi = \epsilon^2 \bar{\bar{p}}_1$, $S = \epsilon \bar{\bar{q}}_2$, $\theta = \bar{\bar{q}}_1 + \lambda(\psi, S)$. After N steps we conclude

$$H = H_N(\epsilon^2 \tilde{p}_1, \tilde{q}_1, \tilde{p}_2, \epsilon \tilde{q}_2, \tilde{p}_3^2 + \tilde{q}_3^2, \epsilon) + O(\epsilon^N) \quad (25)$$

We infer from Hamilton's equations that

$$(d/dt)(\tilde{p}_3^2 + \tilde{q}_3^2) = O(\epsilon^N) ,$$

so that $\tilde{p}_3^2 + \tilde{q}_3^2$ is an adiabatic invariant which changes only by a small amount over very long time intervals. For very long time intervals we may replace $\tilde{p}_3^2 + \tilde{q}_3^2$ by the constant 2μ , which, to lowest order, is twice the usual magnetic moment. We shall employ only the lowest order form (23) although we need to know that in principle a result such as (25) is possible, where H_N is, to lowest order in ϵ , the form given in (23). Clearly over the time interval $O(\epsilon^{-N})$, $\Delta(\tilde{p}_3^2 + \tilde{q}_3^2) = O(\epsilon^{N-M})$, and we might pick N and M as we need for a given situation.

Since we may consider $\tilde{p}_3^2 + \tilde{q}_3^2$ as a constant over some long time interval we have reduced the Hamiltonian in (25) to depend on the two pair of canonically conjugate variables $(\tilde{p}_1, \tilde{q}_1)$ and $(\tilde{p}_2, \tilde{q}_2)$. Further the Hamiltonian depends on these variables in the form

$$H = H(\epsilon^2 \tilde{p}_1, \tilde{q}_1, \tilde{p}_2, \epsilon \tilde{q}_2, \epsilon)$$

so that again there is a separation of time scales with $(\epsilon^2 \tilde{p}_1, \tilde{q}_1)$ vaying on the $O(1/\epsilon^2)$ time scale while $(\tilde{p}_2, \epsilon \tilde{q}_2)$ vary on the $O(1/\epsilon)$ scale. Thus, we have again the possibility of the construction of a second adiabatic invariant. In order to construct the invariant we need not only the time scale separation, but also periodic orbits in the fast variables. For example, we started the construction of the magnetic moment with the observation that the orbits in the (p_3, q_3) plane were ellipses, a property which automatically gives periodic orbits in these variables with all the other variables held fixed. We

must now investigate the implications of the periodicity condition in greater detail for the two distinct cases.

We may introduce the more or less standard guiding center Hamiltonian

$$H_0 = \mu B(\epsilon^2 p_1, \epsilon q_2) + (1/2) G(\epsilon^2 p_1, \epsilon q_2) \times (p_2 - A_S(\epsilon^2 p_1, \epsilon q_2, q_1 + \lambda(\epsilon^2 p_1, \epsilon q_2)))^2 \\ \times B_p^2(\epsilon^2 p_1, \epsilon q_2) / B^2(\epsilon^2 p_1, \epsilon q_2) \quad (26)$$

so that the full Hamiltonian is

$$H = H_0 + \epsilon H_1(\epsilon^2 p_1, q_1, p_2, \epsilon q_2, \epsilon) ; \quad (27)$$

and we shall see that H_1 does not materially alter our conclusions. If we hold $\epsilon^2 p_1$ and q_1 fixed we see that the motion in the $p_2, \epsilon q_2$ plane allows two distinct classes of orbits. In the first class ϵq_2 , which to leading order is S , is bounded above and below. These orbits represent particles trapped by maxima or minima in B and the orbits represent the trapped particles which do not encircle the magnetic axis. The orbits are often called banana orbits as their shape approximates a banana in the $p_2, \epsilon q_2$ plane. Since ϵq_2 is bounded, p_2 is also bounded and the orbits are automatically periodic in the $(p_2, \epsilon q_2)$ planes. For the second class of orbits ϵq_2 increases or decreases monotonically in time and the orbit is untrapped and the particles are often called passing particles. These particles encircle the magnetic axis. We examine the trapped particles first.

When we introduced the magnetic moment adiabatic invariant we

could carry out the process explicitly since the Hamiltonian to any order in ϵ was a polynomial in p_3 and q_3 . There is, however, an implicit process to carry out the adiabatic invariant construction as given by Gardner. For our Hamiltonian (26) and (27) let us first hold $\epsilon^2 p_1$ and q_1 fixed and consider the $p_2, \epsilon q_2$ plane alone. We may construct canonical transformations in the plane starting from the characterization of a canonical transformation as an area preserving transformation. We introduce the area inside the curve $H = \text{const.}$, or equivalently the action,

$$J = \int_{H=\text{const.}} p_2 d\epsilon q_2 \quad (28)$$

and the associated angle variable θ such that area is preserved

$$dp_2 d\epsilon q_2 = dJ d\theta.$$

Such a transformation is canonical and there is a generator G such that

$$p_2 d\epsilon q_2 = -\theta dJ + dG(\epsilon q_2, J).$$

Now G in fact is a function of $\epsilon^2 p_1, q_1$ also so that

$$G = G(\epsilon q_2, J, \epsilon^2 p_1, q_1)$$

and we may then construct a canonical transformation in the full space by the use of the generator

$$F = G(\epsilon q_2, J, \epsilon^2 \bar{p}_1, q_1)/\epsilon + \bar{p}_1 q_1.$$

For such a transformation

$$\epsilon^2 p_1 = \epsilon^2 \bar{p}_1 + O(\epsilon)$$

$$q_1 = \bar{q}_1 + O(\epsilon)$$

and to leading order (p_1, q_1) and $(\bar{p}_1, \bar{q}_1)$ are interchangeable. Further, by construction

$$H = H(J, \epsilon^2 \bar{p}_1, \bar{q}_1) + O(\epsilon) . \quad (29)$$

If we now iterate the transformation in the usual manner we find that after N steps

$$H = H(J^{(N)}, \epsilon^2 p_1^{(N)}, q_1^{(N)}) + O(\epsilon^N) , \quad (30)$$

and J is constant to some very high order in ϵ . Again, for our purposes it suffices to know that (30) is possible, we use only (29).

For the trapped particles we see from (26)

$$p_2 = A_S \pm (2(H - \mu B + O(\epsilon))/G)^{1/2} B/B_p \quad (31)$$

where the + occurs for ϵq_2 increasing and the - sign for ϵq_2 decreasing. The adiabatic invariant to lowest order in ϵ is then

$$J = \int_{S_{\min}}^{S_{\max}} (A_S + ((2H - 2\mu B)/G)^{1/2} B/B_p) dq_2 \\ + \int_{S_{\max}}^{S_{\min}} (A_S - ((2H - 2\mu B)/G)^{1/2} B/B_p) dq_2$$

or

$$J = 2 \int_{S_{\min}}^{S_{\max}} (2(H - \mu B)/G)^{1/2} B/B_p dq_2 \quad (32)$$

In (31), B_p , B , and G are functions of $\varepsilon^2 p_1$ and εq_2 only; there is no q_1 dependence. If invert (31) to obtain H we find that the Hamiltonian corresponding to (30) is

$$H = H_0(\mu, J, \varepsilon^2 p_1) + O(\varepsilon) , \quad (33)$$

so that if we now apply the property that $H = \text{const.}$ on the trajectory we see that $\varepsilon^2 p_1 = \text{const.} + O(\varepsilon)$ for an arbitrarily long time. Since $\varepsilon^2 p_1$ is the flux coordinate of the particle we conclude the particle flux coordinate is bounded within a small interval for a very long time. Hence any trapped particle drifts are essentially nonadiabatic in their very nature. At the end of this section we comment on this point further.

As a preliminary to the study of the passing particles, we must examine the vector potential further. We may normalize the function $S(r, z)$, see (7), so that all flux surfaces $\psi = \text{constant}$ corresponds to the range of values $0 \leq S \leq 1$. Hence, B considered as a function of ψ , S , and θ is periodic of period 1 in S and periodic of period 2π in θ . As follows from (4), $\lambda(r, z)$ is not single valued but if we consider $\lambda = \lambda(\psi, S)$ then

$$\lambda(\psi, S+1) = \lambda(\psi, S) + \Lambda(\psi) . \quad (34)$$

If we consider A_S as a function of ψ , S , and θ and introduce the variables $\psi = \varepsilon^2 p_1$, $S = \varepsilon q_2$, $\theta = q_1 + \lambda(\varepsilon^2 p_1, \varepsilon q_2)$, then

$$\begin{aligned} A_S(\varepsilon^2 p_1, \varepsilon q_2 + n, q_1 + \lambda(\varepsilon^2 p_1, \varepsilon q_2 + n)) \\ = A_S(\varepsilon^2 p_1, \varepsilon q_2, q_1 + \lambda(\varepsilon^2 p_1, \varepsilon q_2) + n\Lambda(\varepsilon^2 p_1)) \end{aligned} \quad (35)$$

Thus, the vector potential is not periodic in ϵq_2 of period 1 unless $n\Lambda(\epsilon^2 p_1)$ is an integral multiple of 2π . We see that we must consider the effects of resonances of the motion the short way around the torus and the motion the long way around the torus.

Let us consider first the simplest case in which $\Lambda(\psi)$ is independent of ψ and there is a positive integer N such that $N\Lambda$ is an integral multiple of 2π ; further let N be the smallest such positive integer. It then follows that in the $(p_2, \epsilon q_2)$ plane the orbit is periodic if ϵq_2 covers the range $0 \leq \epsilon q_2 \leq N$. We may then apply analysis analogous to the trapped particle case and if we take, without loss of generality, the + sign in (31) we find the lowest order adiabatic invariant

$$J = \int_0^N d\epsilon q_2 \{ A_S(\epsilon^2 p_1, \epsilon q_2, q_1 + \lambda(\epsilon^2 p_1, \epsilon q_2)) + (2(H - \mu B)/G)^{1/2} B/B_p \} \quad (36)$$

which we may invert to find the drift hamiltonian

$$H = H(\mu, J, \epsilon^2 p_1, q_1) + O(\epsilon) . \quad (37)$$

We see that the drift Hamiltonian contains q_1 nontrivially so that the trajectory $H = \text{const.}$ admits the possibility of large long term drifts. Thus, the nonaxisymmetric fields introduce the possibility of large, nontrivial drifts. We may readily show, however, that the drifts need not be large. Suppose $N\Lambda = 2\pi M$, where M and N are relatively prime and N is large, but $N\epsilon \ll 1$. In this case

$$\int_0^N d\epsilon q_2 A_S(\epsilon^2 p_1, \epsilon q_2, q_1 + \lambda(\epsilon^2 p_1, \epsilon q_2))$$

$$\begin{aligned}
 &= \sum_{r=0}^{N-1} \int_0^1 d\epsilon q_2 A_S(\epsilon^2 p_1, \epsilon q_2, q_1 + \lambda(\epsilon^2 p_1, \epsilon q_2) + r\Lambda) \\
 &= \frac{N}{2\pi} \int_0^1 d\epsilon q_2 \int_0^{2\pi} dq_1 A_S(\epsilon^2 p_1, \epsilon q_2, q_1 + \lambda(\epsilon^2 p_1, \epsilon q_2)) + O(1) ,
 \end{aligned}$$

or with (6)

$$\int_0^N d\epsilon q_2 A_S(\epsilon^2 p_1, \epsilon q_2, q_1 + \lambda(\epsilon^2 p_1, \epsilon q_2)) = O(1) , \quad (38)$$

where the $O(1)$ estimate in (38) estimates the error as N becomes large.

The adiabatic invariant (36) would then become

$$J = N \int_0^1 d\epsilon q_2 (2(H - \mu B)/G)^{1/2} B/B_p + O(1) , \quad (39)$$

so that for N large and ϵ small the drifts would again become bounded and small, since q_1 is missing in the leading order term in (39).

We may relax slightly the constraint that Λ be constant and $\Lambda/2\pi$ a rational number. We make the slightly weaker assumption that $\Lambda/2\pi$ be close to a rational; specifically $-\epsilon < \Lambda/2\pi - M/N < \epsilon$. In this case we can readily add an $O(\epsilon)$ correction to H_0 and subtract it from H_1 in (20), (27) so that we can take $\Lambda \equiv 2\pi M/N$. Since none of our results depends on H_1 , we see that we may apply the results just given if $\Lambda(\psi)$ remains close (in ϵ) to a given rational number.

When $\Lambda'(\psi)$ is not small the adiabatic invariant construction fails. Although there may be approximately periodic orbits, there is not the nested family of periodic orbits necessary to make the construction work. One can see from the results just obtained that one would expect rapid particle drift away from surfaces where $\Lambda(\psi)/2\pi$ is

approximated by a rational number with small denominator while the drifts would be very slow, if any at all, near surfaces where the rational approximation had a large denominator. Clearly such a process does not lend itself to a continuous, let alone differentiable, description. It is apparent that without the adiabatic invariant we have the real possibility of a large drift formed from very long time cumulation of small drifts.

We note that a standard canonical transformation makes the whole process relatively easy to compute. Since the Hamiltonian H is time independent, $H = \text{const.}$, and we can eliminate two of the variables.⁽⁶⁾ For the general system with two pair of canonically conjugate variables,

$$p_1 dq_1 + p_2 dq_2 - H dt = -d(Ht) + t dH + p_1 dq_1 - (-p_2) dq_2,$$

so that if we invert $H = \text{const.}$ to yield $-p_2 = -p_2(p_1, q_1, H, q_2)$ we may consider $-p_2$ as the "Hamiltonian" of the motion, q_2 as the "time" and (p_1, q_1) and (t, H) as the two pair of canonically conjugate variables. We thus find

$$\begin{aligned} \frac{dH}{dq_2} &= 0, & \frac{dt}{dq_2} &= \frac{\partial q_2}{\partial H} \\ \frac{dp_1}{dq_2} &= \frac{\partial p_2}{\partial q_1}, & \frac{dq_1}{dq_2} &= -\frac{\partial p_2}{\partial p_1} \end{aligned}$$

and in our case from (31) and with the reintroduction of physical variables ψ, S, θ

$$\begin{aligned}
 \frac{d\psi}{dS} &= \varepsilon A_{S,\theta}(\psi, S, \theta + \lambda(\psi, S)) \\
 \frac{d\theta}{dS} &= -\varepsilon \{ A_{S,\psi}(\psi, S, \theta + \lambda(\psi, S)) + \lambda_{,\psi} A_{S,\theta}(\psi, S, \theta + \lambda(\psi, S)) \\
 &\quad + \frac{\partial}{\partial \psi} \left((2(H - \mu B(\psi, S))/G(\psi, S))^{1/2} B(\psi, S)/B_p(\psi, S) \right) \}
 \end{aligned} \tag{40}$$

The system (40) is reminiscent of a Mathieu or Hill's equation for which resonances and unbounded orbits may or may not occur. The unbounded orbits may appear even in cases where adiabatic invariants exist to all orders, just as for the Mathieu equation. Finally it should be noted that if one must follow the orbits for very long times, $\varepsilon \Delta S \gg 1$, then a higher order guiding center Hamiltonian is needed to give meaningful results. We could apply the system (40) to trapped particles as well although at the turning points of the trapped particles (40) becomes singular. It is by no means obviously correct that nonadiabatic effects arising from (40) are reliable when (40) itself is based on the magnetic moment adiabatic invariant.

IV. The Guiding Center Expansion for B_p Small

If we examine the Hamiltonian obtained for the guiding center motion in the general case (26), we see that as B_p becomes small the Hamiltonian becomes badly degenerate. We reexamine the entire problem for $B_p \sim \epsilon$. The coordinates introduced in the previous section (16), (17) are not appropriate to describe the motion in this case. We introduce a new set of coordinates appropriate to the present scaling and consistent with axial symmetry, if present. We are again able to decompose the motion into motions on three distinct time scales, the fastest of which again is the gyro rotation about the guiding center. In contrast with the previous section in which p_θ varied on the slowest time scale, now p_θ varies on the intermediate time scale. Before, the flux coordinate of a particle was essentially p_θ and was canonically conjugate to the toroidal angle coordinate. Now, the flux is conjugate to another coordinate characterizing a given field line. The drift surfaces are now large orbits with or without axial symmetry. A second adiabatic invariant always exists, but the non-axisymmetric magnetic fields have only slight effects on the drift orbits by reconnecting orbits near critical points in the appropriate phase plane.

We recall that if we modify the θ component of (8) to become

$$A_\theta^1 = \psi(\epsilon r, \epsilon z)/(\epsilon r) \quad (8a)$$

then B_T as given by (9) is unaffected but

$$B_p = \epsilon (-\hat{r} \psi_{,\epsilon z}/(\epsilon r) + \hat{z} \psi_{,\epsilon r}/(\epsilon r)) \quad (9a)$$

and

$$H = \frac{1}{2} \{ (p_r - \lambda \psi_{,\epsilon r}/\epsilon - A_r)^2 + (p_z - \lambda \psi_{,\epsilon z}/\epsilon - A_z)^2 + (\epsilon p_\theta - \psi)^2 / (\epsilon r)^2 \} \quad (14a)$$

We introduce a generator of a canonical transformation essentially equivalent to the introduction of (α, β, s) coordinates of the toroidal field alone

$$F = \theta p_1 + (\psi/\epsilon) p_2 + (\lambda/\epsilon) p_3 - p_2 p_3 . \quad (41)$$

from which it follows that

$$\begin{aligned} \theta &= q_1 \\ \psi &= \epsilon q_2 + \epsilon p_3 \\ \lambda &= \epsilon p_2 + \epsilon q_3 \end{aligned} \quad (42)$$

and

$$\begin{aligned} p_\theta &= p_1 \\ p_r - \lambda \psi_{,\epsilon r}/\epsilon &= -q_3 \psi_{,\epsilon r} + p_3 \lambda_{,\epsilon r} \\ p_z - \lambda \psi_{,\epsilon z}/\epsilon &= -q_3 \psi_{,\epsilon z} + p_3 \lambda_{,\epsilon z} \end{aligned} \quad (43)$$

Just as in the previous section the condition that γ be of order 1 in ϵ leads to the conclusion that q_3 and p_3 are both $O(1)$. Since we wish ψ and λ to span a range of values $O(1)$ we infer that ϵq_2 and ϵp_2 must be $O(1)$, and since V_θ must be $O(1)$ we conclude ϵp_1 be $O(1)$. The Hamiltonian (14a) becomes

$$\begin{aligned} H &= (1/2) \{ E(\psi, \lambda) (q_3 + A_\psi(\psi, \lambda, \theta))^2 \\ &\quad + 2F(\lambda, \theta) (q_3 + A_\psi(\psi, \lambda, \theta)) (p_3 - A_\lambda(\psi, \lambda, \theta)) \\ &\quad + G(\lambda, \theta) (p_3 - A_\lambda(\psi, \lambda, \theta))^2 + R(\lambda, \theta) (\epsilon p_1 - \epsilon q_2 - \epsilon p_3)^2 \} \end{aligned} \quad (44)$$

where ψ, λ, θ in (44) are replaced by (42) and

$$\begin{aligned}
 E(\psi, \lambda) &= (\psi_{,\epsilon r})^2 + (\psi_{,\epsilon z})^2 \\
 F(\psi, \lambda) &= -(\psi_{,\epsilon r} \lambda_{,\epsilon r} + \psi_{,\epsilon z} \lambda_{,\epsilon z}) \\
 G(\psi, \lambda) &= (\lambda_{,\epsilon r})^2 + (\lambda_{,\epsilon z})^2 \\
 R(\psi, \lambda) &= 1/(\epsilon r)^2 \\
 \hat{r}A_r + \hat{z}A_z &= A_\psi(\psi, \lambda, \theta) \nabla\psi/\epsilon + A_\lambda(\psi, \lambda, \theta) \nabla\lambda/\epsilon
 \end{aligned} \tag{45}$$

and the underlying structure is

$$H = H(\epsilon p_1, q_1, \epsilon p_2, \epsilon q_2, p_3, q_3, \epsilon) . \tag{46}$$

For a Hamiltonian of this structure, (46),

$$\begin{aligned}
 \dot{\epsilon p_1} &= -\epsilon \partial H / \partial q_1 , & \dot{q_1} &= \epsilon \partial H / \partial (\epsilon p_1) , \\
 \epsilon \dot{p_2} &= -\epsilon^2 \partial H / \partial (\epsilon q_2) , & \epsilon \dot{q_2} &= \epsilon^2 \partial H / \partial \epsilon p_2 , \\
 \dot{p_3} &= -\partial H / \partial q_3 , & \dot{q_3} &= \partial H / \partial p_3 ,
 \end{aligned}$$

so that (p_3, q_3) vary on the fastest $O(1)$ time scale, $(\epsilon p_1, q_1) \equiv (\epsilon p_\theta, \theta)$ varies on the next slower $O(1/\epsilon)$ time scale, while $(\epsilon p_2, \epsilon q_2)$ vary on the slowest $O(1/\epsilon^2)$ time scale. We see that the variable structure is entirely different from the previous section. It is still true that (p_3, q_3) are essentially the components of the gyro velocity, but the other variables are completely rearranged. In particular, p_1 is still p_θ , but it is no longer identified with magnetic flux, rather p_θ is essentially $v_\parallel$. The canonically conjugate coordinate q_1 is essentially θ . The variable λ did not occur before (indeed in the previous section $\lambda \equiv 0$ corresponding to a mirror machine or pure z pinch is admissible). It is still true that ψ varies on the slowest time scale, but its

canonically conjugate coordinate switches from the toroidal angle θ to the field line coordinate λ .

We may now start the construction of the magnetic moment adiabatic invariant and guiding center motion Hamiltonian. We introduce the generator

$$F = (\alpha \bar{p}_3 + A_\lambda)(q_3 + A_\psi) + \beta \bar{p}_3^2/2 + \bar{p}_1 q_1 + \bar{p}_2 q_2 \quad (47)$$

where

$$\alpha^2 = E/B$$

$$\beta = F/B$$

$$B = B_T = (EG - F^2)^{1/2}$$

and all functions of space in (47), (48) are evaluated at ψ, λ, θ followed by the replacements $\psi = \epsilon \bar{p}_2$, $\lambda = \epsilon q_2$ and $\theta = q_1$. We readily find that

$$\epsilon p_1 = \epsilon \bar{p}_1 + O(\epsilon)$$

$$q_1 = \bar{q}_1$$

$$\epsilon p_2 = \epsilon \bar{p}_2 + O(\epsilon)$$

$$\epsilon q_2 = \epsilon \bar{q}_2 + O(\epsilon)$$

$$p_3 = \alpha \bar{p}_3 + A_\lambda + O(\epsilon)$$

$$q_3 = \bar{q}_3/\alpha - \beta \bar{p}_3/\alpha - A_\psi - O(\epsilon) \text{ and}$$

$$H = (1/2)\{B(\bar{p}_3^2 + \bar{q}_3^2) + R(\epsilon \bar{p}_1 - \epsilon \bar{q}_2)^2\} + O(\epsilon) .$$

If we iterate the transformation we obtain

$$H = H_N(\epsilon p_1, q_1, \epsilon p_2, \epsilon q_2, p_3^2 + q_3^2, \epsilon) + O(\epsilon^N)$$

so that we may take $2\mu = p_3^2 + q_3^2$ and we find

$$H = \mu B(\epsilon p_2, \epsilon q_2) + (1/2)R(\epsilon p_2, \epsilon q_2)(\epsilon p_1 - \epsilon q_2)^2 \\ + \epsilon H_1(\epsilon p_1, q_1, \epsilon p_2, \epsilon q_2, \mu, \epsilon) + O(\epsilon^N), \quad (49)$$

and for the most part we need only H_0 .

We may readily construct the second or longitudinal adiabatic invariant after we note that in (49) $(\epsilon p_1, q_1)$ vary on the $O(1/\epsilon)$ scale while $(\epsilon p_2, \epsilon q_2)$ vary on the $O(1/\epsilon^2)$ time scale. We have no difficulties with periodicity or resonance problems as H and the orbits in the $(\epsilon p_1, q_1)$ plane are clearly periodic of period 2π in q_1 . An argument parallel to that given in the previous section shows the existence of the second adiabatic invariant

$$J = (1/2\pi) \int_0^{2\pi} \epsilon p_1 dq_1 + O(\epsilon) = \epsilon p_1 + O(\epsilon). \quad (50)$$

Clearly in the axisymmetric case the adiabatic invariant J becomes the exact constant of the motion p_1 . The Hamiltonian for the remaining drift motion then becomes

$$H = \mu B(\epsilon p_1, \epsilon q_2) + (1/2)R(\epsilon p_2, \epsilon q_2)(J - \epsilon q_2)^2 + O(\epsilon). \quad (51)$$

Since $H = \text{const.}$ on the orbit we may identify the drift orbit, to leading order in ϵ as

$$H_0 = \mu B(\lambda, \psi) + (1/2)R(\lambda, \psi)(J - \lambda)^2 = \text{const.} \quad (52)$$

and we may map out the drift surfaces from (52) in the ψ, λ plane. We see that the drift surfaces are not localized on a surface $\psi = \text{const.}$, but particles may drift far across such surfaces. These drifts occur in the axisymmetric case. The principal effect of the non-axisymmetric

fields is to reconnect the orbits through the critical points $0 = H_{0,\lambda}$
 $= H_{0,\psi}$. We see that the orbits in this case are quite different from
those of the preceding section.

V. An Intermediate Guiding Center Expansion

The results of the two previous sections in which we took $B \sim 1$ but $B_p \sim 1$ or $B_p \sim \epsilon$ are so dissimilar that it is of interest to consider an intermediate case. In this section we take $B_T \sim 1$, $B_p \sim \sqrt{\epsilon}$ are $B_{NA} \sim \epsilon$. Such an ordering may well apply to many Tokamak devices in which $B_p/B_T < 1$ but it may not be quite as small as $O(\epsilon)$. We readily develop the magnetic moment adiabatic invariant series and the associated guiding center Hamiltonian. We need the guiding center Hamiltonian correct through $O(\epsilon)$ and we give most of the details of this development in the Appendix. We find, however, that there is no second adiabatic invariant. The structure of the second invariant is so different in the two cases just considered that it should be no surprise that none exists in the transitional case studied here. We examine the question with two distinct sets of canonically conjugate variables. With the first set, similar to the variables of Sec. III, we find what looks like a second adiabatic invariant, but a careful examination indicates that the "invariant" is constant over a large but limited time interval which is too small to see particle drifts. With the second set of variables, similar to those of Sec. IV, another picture appears. The Hamiltonian contains certain variables which have a rapid oscillation of small amplitude. Thus, no second invariant is possible. But for this situation we present a more or less conventional perturbation expansion of the trapped particle orbits and we determine the flux surface on which the particle lies as a function of time. We show that there is a change in the flux surface at the two tips of the trapped orbit and we give a formula for the shift. We also

study the net change in the flux surface after passing through many orbit tips and we examine some resonant and non-resonant phenomena. The passing particles are much simpler to treat and we also comment briefly on their drifts across the flux surfaces.

We again modify the θ component of the vector potential A' , (8)

$$A'_\theta = \psi(\epsilon r, \epsilon z)/(\epsilon^{3/2} r) \quad (8b)$$

so that B_T is unaffected, but

$$B_p = \sqrt{\epsilon} (-\hat{r} \psi_{,\epsilon z}/(\epsilon r) + \hat{z} \psi_{,\epsilon r}/(\epsilon r)) \quad (9b)$$

and

$$H = (1/2) \{ (p_r - \lambda \psi_{,\epsilon r}/\epsilon - A_r)^2 + (p_z - \lambda \psi_{,\epsilon z}/\epsilon - A_z)^2 + (\epsilon p_2 - \psi/\sqrt{\epsilon})^2/(\epsilon r)^2 \}. \quad (14b)$$

We introduce the generator

$$F = (\psi/\epsilon - \sqrt{\epsilon} p_1) p_2 + \lambda p_3/\epsilon - p_2 p_3 + p_1 \theta \quad (53)$$

so that

$$\begin{aligned} \psi &= \epsilon^{3/2} p_1 + \epsilon q_2 + \epsilon p_3 \\ \lambda &= \epsilon p_2 + \epsilon q_3 \\ \theta &= q_1 + \sqrt{\epsilon} p_2 \end{aligned} \quad (54)$$

and (43) holds for the connection between new and old momenta. If we introduce E , F , G , R , A_ψ and A_λ according to (45) but replace ψ , λ , θ by (54) then we find

$$H = (1/2) (E(q_3 + A_\psi)^2 + 2F(q_3 + A_\psi)(p_3 - A_\lambda) + G(p_3 - A_\lambda)^2)$$

$$+ (1/2) R(\sqrt{\epsilon} q_2 + \sqrt{\epsilon} p_3)^2. \quad (55)$$

We see that the scaling of our variables is now $\epsilon^{3/2} p_1 = O(1)$, $\epsilon p_2 = O(1)$, $\sqrt{\epsilon} q_2 = O(1)$, and both $p_3, q_3 = O(1)$. We have not scaled q_1 , but since A_ψ, A_λ are periodic of period 2π in θ and q_1 , the magnitude of q_1 does not affect the other variables. We now have

$$H = H(\epsilon^{3/2} p_1, q_1, \epsilon p_2, \sqrt{\epsilon} q_2, p_3, q_3, \epsilon)$$

and both $(\epsilon^{3/2} p_1, q_1)$ and $(\epsilon p_2, \sqrt{\epsilon} q_2)$ are on the same, slow, $O(\epsilon^{-3/2})$ time scale while (p_3, q_3) are on the $O(1)$ time scale.

In the appendix we present the details of the canonical transformations which introduce the magnetic moment adiabatic invariant and we show that the transformed Hamiltonian has the structure

$$\begin{aligned} H = & H_0(\psi, \lambda, \sqrt{\epsilon} \hat{q}_2, \epsilon) + \epsilon H_2(\psi, \lambda, \sqrt{\epsilon} \hat{q}_2, \theta, \epsilon) \\ & + \epsilon^{3/2} H_3(\psi, \lambda, \sqrt{\epsilon} \hat{q}_2, \theta, \epsilon) + \dots \end{aligned} \quad (56)$$

where ψ, λ, θ are replaced according to (A 22) and we define

$$\begin{aligned} H_0 = & \mu(B + \epsilon RE/(2B)) + (1/2)(\sqrt{\epsilon} \hat{q}_2)^2 (R - \epsilon R^2 E/B) \\ & - \epsilon R(\sqrt{\epsilon} \hat{q}_2) \langle A_\lambda A_{\psi, \theta} \rangle \end{aligned} \quad (57)$$

$$H_2 = \mu B^2 \left((A_\psi/B)_{,\lambda} - (A_\lambda/B)_{,\psi} \right),$$

and according to (A 23) we may consider the coordinate of the flux surface on which the particle moves to be $\epsilon^{3/2} \hat{p}_1 + \epsilon \hat{q}_2$. We have scaled time according to the period for a cyclotron oscillation. Since we have eliminated this oscillation from the Hamiltonian it is

convenient to rescale time and the variables according to the drift time scale, which as we just saw is $O(\epsilon^{-3/2})$. If we set $\tau = \epsilon^{3/2}t$ we see tha

$$\begin{aligned} \hat{p}_1 d\hat{q}_1 + \hat{p}_2 d\hat{q}_2 - H dt \\ = \epsilon^{-3/2} ((\epsilon^{3/2} \hat{p}_1) d\hat{q}_1 + (\epsilon \hat{p}_2) d(\sqrt{\epsilon} \hat{q}_2) - H d\tau), \end{aligned}$$

so that a rescaling canonical transformation would be $\epsilon^{3/2} \hat{p}_1 = p_1$,

$\hat{q}_2 = q_1$, $\epsilon \hat{p}_2 = p_2$, $\sqrt{\epsilon} \hat{q}_2 = q_2$, and

$\tau = \epsilon^{3/2} t$ and with this transformation and the definition

$$\eta = \sqrt{\epsilon}, \quad (58)$$

$$H = H_0(\psi, \lambda, q_2, \epsilon) + \eta^2 H_2(\psi, \lambda, q_2, \theta, \epsilon) + \eta^3 H_3(\psi, \lambda, q_2, \theta, \epsilon) + \dots \quad (59)$$

where H_0 and H_2 are still given by (57), but

$$\begin{aligned} \psi &= p_1 + \eta q_2 \\ \lambda &= p_2 \\ \theta &= q_1 + p_2/\eta \end{aligned} \quad (60)$$

We see that the second and higher order terms in the Hamiltonian have rapid dependence on the canonical variable p_2 . It is this rapid dependence on p_2 that destroys the possibility of the existence of an adiabatic invariant.

It is instructive to study the possibility of the existence of a second adiabatic invariant before we proceed with a direct analysis

of (59), (60). To this end we go back to the Hamiltonian (56) and variables (A 23). The question becomes clearer after yet another canonical transformation. If we introduce the variables

$$\begin{aligned}\epsilon P_2 &= -\sqrt{\epsilon} \hat{q}_2 \\ Q_1 &= \hat{q}_1 + \sqrt{\epsilon} \hat{p}_2\end{aligned}\tag{61}$$

$$\begin{aligned}\epsilon P_2 &= -\sqrt{\epsilon} \hat{q}_1 \\ \epsilon Q_2 &= \epsilon^{3/2} \hat{p}_1 + \epsilon \hat{q}_2\end{aligned}$$

then a direct computation of Poisson brackets shows that

$$\{P_i, Q_j\} = \delta_{ij}, \quad \{P_1, P_2\} = \{Q_1, Q_2\} = 0,$$

so that the transformation is indeed canonical, ⁽⁷⁾ and the Hamiltonian is given by

$$\begin{aligned}H &= H_0(\psi, \lambda, \epsilon P_1, \theta, \epsilon) + \epsilon H_2(\psi, \lambda, \epsilon P_1, \theta, \epsilon) \\ &+ \epsilon^{3/2} H(\psi, \lambda, \epsilon P_1, \theta, \epsilon) + \dots\end{aligned}\tag{62}$$

with the substitutions

$$\begin{aligned}\psi &= \epsilon Q_2 \\ \lambda &= \epsilon P_2 + \sqrt{\epsilon} Q_1 \\ \theta &= Q_1\end{aligned}\tag{63}$$

and with H_0, H_2, H_3 , etc. still given by (57). Parenthetically we note that the variables (63) are essentially the same as those of

section IV, compare (42) and (53), while the variables used elsewhere in this section are essentially those of section III, compare (16) and (A 22). With the Hamiltonian (62) we see that (P_1, Q_1) vary on the $O(1/\epsilon)$ time scale, but (P_2, Q_2) vary on the $O(1/\epsilon^2)$ time scale. Thus we have the first prerequisite for the construction of an adiabatic invariant, the separation of time scales. We note that the variables we use for this discussion destroy the axisymmetry of motion in an axisymmetric magnetic field.

In order to proceed with the construction of the adiabatic invariant, we must also find families of periodic orbits. If we examine the leading order Hamiltonian H_0 in (63) we see that the variable $\lambda = \epsilon P_2 + \sqrt{\epsilon} \theta$ occurs. Thus, an orbit does not close on itself until $\sqrt{\epsilon} \theta$ is $O(1)$, or at least $\tau \sim 1/\epsilon^{3/2}$. Even over such a long time interval there will not be a family of periodic orbits but only isolated periodic orbits. Clearly we are back to the study of resonance problems explored at the end of section III. We might view the question in another way. We could expand a function

$f(\lambda) = f(\epsilon P_2 + \sqrt{\epsilon} \theta) = f(\epsilon P_2) + \sqrt{\epsilon} \theta f'(\epsilon P_2) + \dots$. In this way the lowest order Hamiltonian would be independent of $Q_1 \equiv \theta$, and so the orbits would be periodic on an $O(1)$ time scale. We could then easily construct a second adiabatic invariant exactly as in section IV and we would have particles confined on or near flux surfaces. Apparently a second invariant does exist, but let us consider the matter in greater detail. For the expansion just given to hold we require $\sqrt{\epsilon} \theta$ must be small. When $\sqrt{\epsilon} \theta$ is $O(1)$ the representation of the Hamiltonian is poor. Thus, we obtain a second adiabatic invariant, but only over the limited time interval $t \lesssim \epsilon^{-3/2}$. for which $\sqrt{\epsilon} \theta$ is $\lesssim 1$. Over this

interval we might see what looks like an adiabatic invariant, but as time increases, the constancy must fail. We see shortly that the drifts do indeed occur, but on an $O(\epsilon^{-11/4})$ time scale, a result fully consistent with the analysis here. We set aside the possibility of an adiabatic invariant as the invariant we found cannot describe the orbit for a long enough time and we return to the variables and Hamiltonian given by (59) and (60).

Hamilton's equations for the system (59), (60) are

$$-\dot{p}_1 = \eta^2 H_{2,q_1} + \eta^3 H_{3,q_1} + \dots, \quad \dot{q}_1 = H_{0,p_1} - \eta^2 H_{2,p_1} - \eta^3 H_{3,p_1} + \dots \quad (64)$$

$$-\dot{p}_2 = H_{0,q_2} + \eta^2 H_{2,q_2} + \eta^3 H_{3,q_2} + \dots, \quad \dot{q}_2 = H_{0,p_2} + \eta H_{2,q_1} + \eta^2 (H_{3,q_1} + H_{2,p_2}) + \dots$$

where we consider $H = H(p_1 + \eta q_2, p_2, q_2, q_1 + p_2/\eta)$ and by convention $H_{,p_2}$ means a derivative with respect to the second argument only. That part of a partial derivative of H with respect to p_2 coming from differentiation with respect to the fourth argument is represented as a derivative with respect to q_1 . With this convention each term in (64) is of the order of magnitude in η explicitly exhibited. We intend to study the drift motion by examining (64) over a limited time interval and then repeating the process for the next limited time interval. With such an approach we may employ a conventional perturbation expansion in powers of η in (64). We expand in the form

$$(p_i(t), q_i(t)) = (p_i^{(0)}(t, \eta), q_i^{(0)}(t, \eta)) + \eta(p_i^{(1)}(t, \eta), q_i^{(1)}(t, \eta)) + \eta^2(p_i^{(2)}(t, \eta), q_i^{(2)}(t, \eta)) + \dots$$

where terms $(p_i^{(j)}(t, \eta), q_i^{(j)}(t, \eta))$ may be $o(1)$ in η although they are never larger than $O(1)$. To lowest order we find

$$\dot{p}_1^{(0)} = 0 \quad , \quad \dot{q}_1^{(0)} = H_{0,p_1}^{(0)} \quad (65)$$

$$\dot{p}_2^{(2)} = H_{0,q_2}^{(0)} \quad , \quad \dot{q}_2^{(2)} = H_{0,p_2}^{(0)}$$

We may solve the system (65) quite simply since $p_1^{(0)} = \text{const.}$ and q_1 does not appear in H_0 . The motion reduces in the p_2, q_2 plane to the trajectory $H_0(p_1^{(0)} + \epsilon q_1^{(0)}, p_2^{(0)}, q_2^{(0)}) = \text{const.}$ We may again distinguish two classes of orbits, closed curves in the p_2, q_2 plane corresponding to trapped particle orbits, and unbounded curves corresponding to passing particles. The passing particles generate a periodic orbit when one recognizes that according to (34) when $\lambda \equiv p_2$ increases by $\Lambda(\psi)$ the point (ψ, λ, θ) returns to the same physical point. We examine trapped particles in greatest detail, although we give some indication of what we believe is the solution with passing particles.

In the solution of the full system (64) by perturbation techniques based on the lowest order system (65) we must always solve the variational equations associated with the lowest order system. If we hold p_1 constant the variational system is a linear set of differential equations for $p_2(t), q_2(t)$:

$$-\dot{p}_2(t) = H_{0,q_2 p_2}^{(0)} p_2(t) + H_{,q_2 q_2}^{(0)} q_2(t) + f(t) \quad (66)$$

$$\dot{q}_2(t) = H_{0,p_2 p_2}^{(0)} p_2(t) + H_{,q_2 p_2}^{(0)} q_2(t) + g(t),$$

where such a term as $H_{0,q_2 p_2}^{(0)}$ is evaluated at the zeroth order solution of (65). One solution of the homogeneous system associated with (66)

is $(\dot{p}_2^{(0)}(t), \dot{q}_2^{(0)}(t))$ which we denote by $(P(t), Q(t))$. We introduce a second solution $(p(t), q(t))$ normalized so that

$$P(t)q(t) - Q(t)p(t) = 1 \quad (6)$$

and the solution to the variational system (66) is

$$p_2(t) = P(t)I_1(t) + p(t)I_2(t) \quad (68)$$

$$q_2(t) = Q(t)I_1(t) + q(t)I_2(t)$$

where

$$I_1 = - \int_0^t dt' (f(t')q(t') + g(t')p(t'))$$

and

$$I_2 = \int_0^t dt' (f(t')Q(t') + g(t')P(t'))$$

The first order solution satisfies the system

$$\begin{aligned} -\dot{p}_1^{(1)} &= n H_{2,q_1}^{(0)} + n H_{2,q_1 q_1}^{(0)} (p_2^{(1)}/n) \\ -\dot{p}_2^{(1)} &= H_{0,q_2 p_2}^{(0)} p_2^{(1)} + H_{0,q_2 q_2}^{(0)} q_2^{(1)} + n^2 H_{0,q_2}^{(0)} \\ \dot{q}_2^{(1)} &= H_{0,p_2 p_2}^{(0)} p_2^{(1)} + H_{0,p_2 q_2}^{(0)} q_2^{(1)} \\ &\quad + n H_{2,q_1}^{(0)} + n^2 (H_{2,p_2}^{(0)} + H_{3,q_1}^{(0)}) \end{aligned} \quad (70)$$

The formulas (66)-(70) apply equally well to trapped or untrapped particles.

We now turn our attention specifically to trapped particles and estimate the solution to (70) appropriately for trapped particles. We readily find, following the notation of (68), that

$$I_1 = -n \int_0^t dt' H_{2,q_1}^{(0)}(t') p(t') + o(n^2)$$

and

$$I_2 = o(n^2)$$

so that

$$\begin{aligned} -p_1^{(1)}(t) &= n \int_0^t dt' H_{2,q_1}^{(0)}(t') \\ &\quad - n \int_0^t dt' H_{2,q_1 q_1}^{(0)}(t') P(t') \cdot \int_0^{t'} dt'' H_{2,q_1}^{(0)}(t'') p(t'') + o(n^2). \end{aligned}$$

From the identity

$$n (d/dt) H_{2,q_1}^{(0)} = H_{0,q_1 q_1}^{(0)} P(t) + O(n)$$

we readily conclude

$$-p_1^{(1)} = \int_0^{t'} dt'' H_{2,q_1}^{(0)}(t'') p(t'') + o(n^2).$$

so that the shift in the flux surface as a function of time is

$$\begin{aligned} \Delta \psi(t) &= -n^2 \int_0^t dt' H_{2,q_1}^{(0)}(t') \\ &\quad - n^2 Q(t) \int_0^t dt' H_{2,q_1}^{(0)}(t') p(t) + o(n^3). \end{aligned}$$

We have not yet specified the point $t = 0$ or the time interval over which we evaluate $\Delta \psi(t)$. We obtain a particularly simple and symmetric representation if we set $t = 0$ to be the point at which $\dot{q}_2^{(0)}(t) = 0$, or $Q(t=0) = 0$. We let $T(\psi)$ be the period for this orbit and we then find

$$\Delta \psi(T) = -n^2 \int_0^T dt' H_{2,q_1}^{(0)}(t') + o(n^3). \quad (71)$$

We examine (71) in somewhat greater detail.

The trapped particle orbit $(p_2^{(0)}(t), q_2^{(0)}(t))$ is a closed curve in the (p_2, q_2) plane. We define the tips of the orbit as the points at which $p_2(t) \equiv \lambda$ assumes the largest and smallest values. Clearly these are points at which $P(t)$ vanishes. Equally, the points of stationary $V_{\parallel} \equiv q_2(t)$ are the points at which $Q(t) = 0$. During one period the toroidal angle θ increases by an amount

$$\Delta \theta = \int_0^{T(\psi)} dt H_{0,p_1}^{(0)} \quad (72)$$

Let t_1 and t_2 be the times of the two consecutive banana tips, $0 < t_1 < t_2 < T$, so that by symmetry

$$q_1^{(0)}(t_2) - q_1^{(0)}(t_1) = \Delta \theta / 2$$

Let us examine the essential element in (71)

$$I = \int_0^T dt H_{2,q_1}(p_1^{(0)} + n q_2^{(0)}(t), p_2^{(0)}(t), q_2^{(0)}(t), q_1^{(0)}(t) + p_2^{(0)}(t)/n) , \quad (73)$$

Except in the neighborhood of the points t_1, t_2 where $P(t) = 0$ we may use

$$H_{2,q_1} = n((d/dt)H_2)/P(t) + O(n)$$

and integrate by parts to conclude that $I = O(n)$. Hence, the leading order terms in (73) come entirely from the stationary points of the integrand t_1 and t_2 . We are led to perform a stationary phase evaluation of (73). If we expand about an orbit tip $t = t_i$, $i = 1, 2$ we find

$$p_2^{(0)}(t) = p_2^{(0)}(t_i) + (p_2^{(0)}(t_i)/2)(t-t_i)^2 + \dots \text{ and if we}$$

introduce

$$C_i = p_2^{(0)}(t_i) = - H_{,q_2 q_2}^{(0)}(t_i) H_{,p_2}^{(0)}(t_i), \quad (74)$$

then we readily find

$$\begin{aligned} \Delta \psi = & - n^{5/2} \sqrt{2} \int_0^\infty (dx/\sqrt{x}) \cdot \{ H_{2,q_1}^{(0)}(\psi(t), \lambda(t_1), q_2^{(0)}(t_1), \theta + p_2^{(0)}(t_1)/n + x C_1) \\ & + H_{2,q_1}^{(0)}(\psi(t), \lambda(t_2), q_2^{(0)}(t_2), \theta + \Delta\theta/2 + p_2^{(0)}(t_2)/n + x C_2) \} + o(n^3) \end{aligned} \quad (75)$$

and

$$\Delta \theta = \int_0^{T(\psi)} dt H_{0,p_1}^{(0)}.$$

In the evaluation of (75), H_0 and the orbits $p_2^{(0)}(t)$, $q_2^{(0)}(t)$ are held fixed, so that (75) is a property of the orbit alone. Further, the orbit is characterized by the values of ψ and of H (a constant) alone. Thus, one may view (75) as a discrete process which describes the slow drift of the trapped particle orbits

$$\psi_{n+1} - \psi_n = n^{5/2} f(\psi_n, \theta_n) \quad (76)$$

$$\theta_{n+1} = \theta_n = g(\psi_n)$$

where f and g are given by (75). The asymptotic evaluation of (71) as (75) is identical with the usual stationary phase integration for $H_2(\psi, \lambda, q_2, \theta) = f(\psi, \lambda, q_2) \exp(in \theta)$, $n \neq 0$. We see from (75) that only the neighborhoods of the trapped particle orbit tips contribute in leading order to $\Delta \psi$. The existence of a second adiabatic invariant

would imply that particle drift is a continuous process rather than a discrete one. Thus, (75) is inconsistent with the existence of an adiabatic invariant as commonly understood.

We may show resonance phenomena in the system (75), (76). We examine (76) after N steps where we assume $N\eta^{5/2} \ll 1$ and we find

$$\psi_{N+1} - \psi_1 = \eta^{5/2} \left(\sum_{n=0}^{N-1} f(\psi_1, \theta_1 + n g(\psi_1)) + o(1) \right) \quad (77)$$

If $g(\psi_1)$ is a rational multiple of 2π then $\psi_{N+1} - \psi_1$ is proportional to N , while for $g(\psi_1)$ an irrational multiple of 2π , $\psi_{N+1} - \psi_1$ is proportional to $N\eta^{5/2} \langle f(\psi_1, \theta) \rangle$, which in our case vanishes. Since the process (76) is only accurate to a correction $O(\eta^{7/2})$, we may approximate any irrational with a rational with an error, say $O(\sqrt{\eta})$, and obtain a mean shift $\Delta \psi$. If $f(\psi, \theta)$ contains only one or at most a few harmonics, then the resonance condition is far more stringent and our simple model may not suffice. In this case $g(\psi_1)$ must be well approximated by a rational with a denominator equal to one of the harmonics present in the fourier series representation of $f(\psi, \theta)$. Otherwise the drift approximately vanishes. When we consider the full ψ dependence in (76) rather than the approximation (77), the process becomes more intricate and the possibility of slow drift across flux surfaces occurs.

There is still another mechanism of significant slow drift across flux surface beyond those implied by (76). In many cases the net effect of the discrete process (76) will give a net zero drift, especially when $\langle f \rangle = 0$ or there are only a very few harmonics in the Fourier series expansion of $f(\psi, \theta)$. It is easy to show however that if

one carries the expansion in (71) through fourth order in η , then there are terms in $\Delta \psi$ which do not average to zero and which do give rise to a small but steady $\Delta \psi$ which is $O(\eta^4)$. The actual evaluation of the drift is an exceedingly laborious calculation and we do not have the result. It should be noted that when the full ψ dependence of (76) is used, similar terms $O(\eta^4)$ appear in the discrete process as well. In the strongly resonant case we do not need this effect to describe drifts, but off resonance such terms do produce a determinable drift motion. We are not, however, prepared to explore the matter further.

For the case of passing particles (56) - (71) still hold, but now $P(t) = p_2^{(0)}(t)$ never vanishes so that there is no stationary phase point in the evaluation of (71). Although $p_2^{(0)}(t)$ is not periodic, $p_2^{(0)}(t) - t \Lambda(\psi)$ is periodic so that $P(t)$ is periodic and we find from (71)

$$\Delta \psi(T) = o(\eta^3) . \quad (78)$$

For passing particles there is no discrete process such as (76), although if we extend (71) to higher order in η we find that there is, just as for trapped particles, a higher order small drift which does not average to zero. Its calculation is an intricate matter and we are unprepared to offer detailed results; it is easy to show, however, that $\Delta \psi(T)$ in this case is $o(\eta^4)$, so that the drifts are smaller than for trapped particles.

Appendix A. The Intermediate Scaling Guiding Center Hamiltonian

In Section V we require the Hamiltonian to more than lowest order. Here we give the sequence of canonical transformations leading to the desired form. This appendix gives an explicit example of how we may employ canonical transformations to simplify the problem and obtain higher order corrections in the Hamiltonian. We start from the Hamiltonian (55), with E, F, G, R, A_ψ and A_λ defined by (45) and with the space coordinates ψ, λ, θ replaced by (54). Throughout this treatment we shall consider functions of space coordinates as functions of ψ, λ, θ followed by some replacements of ψ, λ, θ by canonically conjugate coordinates. We start by expanding the various functions of the space coordinate ψ, λ, θ in Taylor series in $\epsilon p_3, \epsilon q_3$ in the form

$$\begin{aligned} f(\psi, \lambda, \theta) &= f(\epsilon^{3/2} p_1 + \epsilon q_2 + \epsilon p_3, \epsilon p_2 + \epsilon q_3, q_1 + \sqrt{\epsilon} p_2) \\ &= f(\epsilon^{3/2} p_1 + \epsilon q_2, \epsilon p_2, q_1 + \sqrt{\epsilon} p_2) \\ &\quad + \epsilon (f_{,\psi} p_3 + f_{,\lambda} q_3) + (\epsilon^2/2) (f_{,\psi\psi} p_3^2 + 2f_{,\psi\lambda} p_3 q_3 + f_{,\lambda\lambda} q_3^2) + \dots \end{aligned}$$

where ψ, λ, θ on the right hand side are replaced by $\psi = \epsilon^{3/2} p_1 + \epsilon q_2$, $\lambda = \epsilon p_2$, and $\theta = q_1 + \sqrt{\epsilon} p_2$. We now start the canonical transformations with the generator

$$\begin{aligned} F &= (\bar{p}_3 + A_\lambda (\epsilon^{3/2} \bar{p}_1 + \epsilon \bar{q}_2, \epsilon \bar{p}_2, q_1 + \sqrt{\epsilon} \bar{p}_2)) \\ &\quad + (q_3 + A_\psi (\epsilon^{3/2} \bar{p}_1 + \epsilon \bar{q}_2, \epsilon \bar{p}_2, q_1 + \sqrt{\epsilon} \bar{p}_2)) + \bar{p}_1 q_1 + \bar{p}_2 q_2, \end{aligned} \quad (A1)$$

from which it follows that

$$\begin{aligned}
 q_1 &= \bar{q}_1 + O(\epsilon^{3/2}) , \quad p_1 = \bar{p}_1 + O(1) \\
 \bar{p}_3 &= p_3 - A_\lambda (\epsilon^{3/2} p_1 + \epsilon q_2, \epsilon p_2, q_1 + \sqrt{\epsilon} p_2) + O(\epsilon^{3/2}) \\
 \bar{q}_3 &= q_3 + A_\psi (\epsilon^{3/2} p_1 + \epsilon q_2, \epsilon p_2, q_1 + \sqrt{\epsilon} p_2) + O(\epsilon^{3/2}) \\
 p_2 &= \bar{q}_2 + O(\epsilon) \\
 \bar{q}_2 &= q_2 + \sqrt{\epsilon} (A_{\lambda, \theta} \bar{q}_3 + A_{\psi, \theta} \bar{p}_3 + A_{\psi, \theta} A_\lambda) + O(\epsilon)
 \end{aligned} \tag{A2}$$

where we may evaluate the potentials at the new or old coordinates. Since we wish to follow the flux surface on which a particle lies we must determine the combination $\epsilon^{3/2} p_1 + \epsilon q_2$ with some care. We see that

$$\epsilon^{3/2} \bar{p}_1 + \epsilon \bar{q}_2 = \epsilon^{3/2} p_1 + \epsilon q_2 + O(\epsilon^2) \tag{A3}$$

and that the error term is just a point function of the coordinates, independent of time and not equivalent to a particle drift across a flux surface. After the transformations

$$\begin{aligned}
 H &= (1/2) \{ (E \bar{q}_3^2 + 2F \bar{q}_3 \bar{p}_3 + G \bar{p}_3^2) \\
 &\quad + R \{ \sqrt{\epsilon} (\bar{q}_2 + A_\lambda) \}^2 \} + \sqrt{\epsilon} \bar{p}_3 R / \epsilon (\bar{q}_2 + A_\lambda) \\
 &\quad + \epsilon T_1 / 2 + O(\epsilon^{3/2}) ,
 \end{aligned} \tag{A4}$$

where

$$\begin{aligned}
 T_1 &= \{ R \bar{p}_3^2 - 2R \sqrt{\epsilon} (\bar{q}_2 + A_\lambda) \} (A_{\lambda, \theta} \bar{q}_3 + A_{\psi, \theta} \bar{p}_3 + A_{\psi, \theta} A_\lambda) \\
 &\quad + (\sqrt{\epsilon} (\bar{q}_2 + A_\lambda))^2 (R_{, \psi} \bar{p}_3 + R_{, \lambda} \bar{q}_3 + R_{, \psi} A_\lambda - R_{, \lambda} A_\psi) \\
 &\quad + (E_{, \psi} A_\lambda - E_{, \lambda} A_\psi + E_{, \psi} \bar{p}_3 + E_{, \lambda} \bar{q}_3) \bar{q}_3^2
 \end{aligned}$$

$$\begin{aligned}
 & + 2(F_{,\psi}A_{,\lambda} - F_{,\lambda}A_{,\psi} + F_{,\psi}\bar{p}_3 + F_{,\lambda}\bar{q}_3)\bar{q}_3\bar{p}_3 \\
 & + (G_{,\psi}A_{,\lambda} - G_{,\lambda}A_{,\psi} + G_{,\psi}\bar{p}_3 + G_{,\lambda}\bar{q}_3)p_3^{-2} \quad (A5) \\
 & + 2E\bar{q}_3(A_{\psi,\psi}A_{\lambda} - A_{\psi,\lambda}A_{\psi} + A_{\psi,\psi}\bar{p}_3 + A_{\psi,\lambda}\bar{q}_3) \\
 & + 2F[-\bar{q}_3(A_{\lambda,\psi}A_{\lambda} - A_{\lambda,\lambda}A_{\psi} + A_{\lambda,\psi}\bar{p}_3 + A_{\lambda,\lambda}\bar{q}_3) \\
 & + \bar{p}_3(A_{\psi,\psi}A_{\lambda} - A_{\psi,\lambda}A_{\psi} + A_{\psi,\psi}\bar{p}_3 + A_{\psi,\lambda}\bar{q}_3)] \\
 & - 2G\bar{p}_3(A_{\lambda,\psi}A_{\lambda} - A_{\lambda,\lambda}A_{\psi} + A_{\lambda,\psi}\bar{p}_3 + A_{\lambda,\lambda}\bar{q}_3)
 \end{aligned}$$

and where the replacement of the functions of space ψ, λ, θ , is $\psi = \epsilon^{3/2} \bar{p}_1 + \epsilon \bar{q}_2$, $\lambda = \epsilon \bar{p}_2$, and $\theta = \bar{q}_1 + \sqrt{\epsilon} \bar{p}_2$.

We now start the construction of the lowest order magnetic moment adiabatic invariant. We see that through order $\sqrt{\epsilon}$ the curves $H = \text{const.}$ in the $\bar{p}_3, \bar{q}_3$ plane are slightly off-center ellipses. An affine transformation will shift and stretch the ellipses to circles and so we introduce the next generator in terms of the as yet unspecified functions $\alpha(\psi, \lambda)$, $\beta(\psi, \lambda)$, $\gamma(\psi, \lambda)$, and $\delta(\psi, \lambda)$

$$\begin{aligned}
 F = & (\alpha\tilde{p}_3 + \sqrt{\epsilon} \gamma(\bar{q}_2 + A_{\lambda})) \cdot (\bar{q}_3 + \sqrt{\epsilon} \delta(\bar{q}_2 + A_{\lambda})) \\
 & + \beta \tilde{p}_3^2/2 + \tilde{p}_1\bar{q}_1 + \tilde{p}_2\bar{q}_2 \quad (A6)
 \end{aligned}$$

where

$$\begin{aligned}
 \psi &= \epsilon^{3/2} \tilde{p}_1 + \epsilon \bar{q}_2 \\
 \lambda &= \epsilon \tilde{p} \\
 \theta &= \bar{q}_1 + \sqrt{\epsilon} \tilde{p}_2 \quad (A7)
 \end{aligned}$$

The relations between the new and old variables are

$$\begin{aligned}
 \bar{p}_1 &= \tilde{p}_1 + O(\sqrt{\epsilon}), & \bar{q}_1 &= \tilde{q}_1 + O(\epsilon^{3/2}), \\
 \bar{p}_2 &= \tilde{p}_2 + O(\sqrt{\epsilon}), & \bar{q}_2 &= \tilde{q}_2 + O(\epsilon) \\
 \bar{p}_3 &= \alpha \tilde{p}_3 + \sqrt{\epsilon} \gamma (\tilde{q}_2 + A_\lambda) \\
 \bar{q}_3 &= -\beta \tilde{p}_3/\alpha + \tilde{q}_3/\alpha - \sqrt{\epsilon} \delta \sqrt{\epsilon} (\tilde{q}_2 + A_\lambda),
 \end{aligned} \tag{A8}$$

while the flux coordinate is essentially unchanged,

$$\epsilon^{3/2} \bar{p}_1 + \epsilon \bar{q}_2 = \epsilon^{3/2} \tilde{p}_1 + \epsilon \tilde{q}_2 + O(\epsilon^2), \tag{A9}$$

with a time independent point function for error, which does not represent a drift. If we introduce the new variables into (A4) and set

$$\begin{aligned}
 B^2 &= E G - F^2 \\
 \alpha^2 &= E/B \\
 \beta &= F/B \\
 \gamma &= -R E/B^2 \\
 \delta &= -R F/B^2
 \end{aligned} \tag{A10}$$

we find

$$\begin{aligned}
 H &= (1/2) (\tilde{p}_3^2 + \tilde{q}_3^2) B + (1/2) R (\sqrt{\epsilon} (\tilde{q}_2 + A_\lambda))^2 \cdot (1 - \epsilon R E/B^2) \\
 &+ (\epsilon T_1)/2 + O(\epsilon^{3/2})
 \end{aligned} \tag{A11}$$

where we make the substitutions

$$\begin{aligned}
 \bar{p}_3 &= \alpha \tilde{p}_3 \\
 \bar{q}_3 &= -\beta \tilde{p}_3/\alpha + \tilde{q}_3/\alpha
 \end{aligned} \tag{A12}$$

in the expression (A5) for T_1 . Functions of space are now evaluated at

$$\psi = \epsilon^{3/2} \tilde{p}_1 + \epsilon \tilde{q}_2, \quad \lambda = \epsilon \tilde{p}_2, \quad \theta = \tilde{q}_1 + \sqrt{\epsilon} \tilde{p}_2.$$

As we need the Hamiltonian to first order in ϵ we must get the

magnetic moment correct through order ϵ . For clarity and simplicity let us consider the following problem in Hamiltonian dynamics in the canonically conjugate set of variables (p, q) ,

$$H = (1/2) A(p^2 + q^2) + \epsilon P(p, q) ,$$

where $P(p, q)$ is a polynomial of degree N in p, q . Suppose we introduce the generator of a transformation near the identity

$$F = \bar{p} q + \epsilon Q(\bar{p}, q) , \quad (A14)$$

where $Q(\bar{p}, q)$ is a polynomial of degree N in $(\bar{p}, q)$, so that

$$\bar{q} = q + \epsilon \frac{\partial Q}{\partial \bar{p}}$$

$$p = \bar{p} + \epsilon \frac{\partial Q}{\partial q} ,$$

or

$$q = \bar{q} - \epsilon \frac{\partial Q(\bar{p}, \bar{q})}{\partial \bar{p}} + O(\epsilon^2)$$

$$p = \bar{p} + \epsilon \frac{\partial Q(\bar{p}, \bar{q})}{\partial \bar{q}} + O(\epsilon^2)$$

and the Hamiltonian becomes

$$H = A (\bar{p}^2 + \bar{q}^2)/2 + \epsilon \left\{ A \left(\bar{p} \frac{\partial Q}{\partial \bar{q}} - \bar{q} \frac{\partial Q}{\partial \bar{p}} \right) + P(\bar{p}, \bar{q}) \right\} + O(\epsilon^2) \quad (A15)$$

We seek to select $Q(\bar{p}, \bar{q})$ so as to remove as much of the $O(\epsilon)$ terms in H as possible. We introduce polar coordinates in the $\bar{p}, \bar{q}$ plane such that $\bar{p} = \rho \cos \theta$, $\bar{q} = \rho \sin \theta$ and we find

$$A \left(\bar{p} \frac{\partial Q}{\partial \bar{q}} - \bar{q} \frac{\partial Q}{\partial \bar{p}} \right) + P(\bar{p}, \bar{q}) = A \frac{\partial Q}{\partial \theta} + P(\rho \cos \theta, \rho \sin \theta) . \quad \text{Thus, if we}$$

express $P(\rho \cos \theta, \rho \sin \theta)$ as a Fourier series in θ , then by an appropriate choice of Q we may eliminate all but the constant terms. Specifically if $P = \bar{p}^m \bar{q}^n$, then we may eliminate the term if $m + n$ is not even, while if $m + n = 2r$, then we may transform $\bar{p}^{2r-n} \bar{q}^n$ into $(\bar{p}^2 + \bar{q}^2)^r \int_0^{2\pi} (\cos \theta)^{2r-n} (\sin \theta)^n d\theta / 2\pi$. We now apply this procedure to our Hamiltonian and we extend the transformation to the full set of variables. We readily find

$$\begin{aligned} H = & (1/2) (\tilde{p}_3^2 + \tilde{q}_3^2) [B + \epsilon \{ RE + 2B^3 \{ (A_\psi/B)_{,\lambda} - (A_\lambda/B)_{,\psi} \} / 2B] \\ & + R(\sqrt{\epsilon}(\tilde{q}_2 + A_\lambda))^2 (1 - \epsilon RE/B^2) \\ & - 2\epsilon R/\epsilon (\tilde{q}_2 + A_\lambda) A_\lambda A_{\psi,\theta} \\ & + \epsilon (\sqrt{\epsilon}(\tilde{q}_2 + A_\lambda))^2 (R_{,\psi} A_\lambda - R_{,\lambda} A_\psi) + O(\epsilon^{3/2}). \end{aligned} \quad (A16)$$

We could, of course, repeat the process to higher order in ϵ to conclude that $\tilde{p}_3^2 + \tilde{q}_3^2$ is constant, say 2μ , to high order. We could work with (A16) but it simplifies the subsequent analysis to make two further transformations.

Let $f(\theta)$ be an arbitrary continuous function of θ , periodic of period 2π and with zero average

$$\langle f(\theta) \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta = 0.$$

It is desirable to introduce an indefinite integral of $f(\theta)$, say If , such that $If(\theta)$ is periodic of period 2π and $\langle If \rangle = 0$. Our integral operator I applied to $\exp(in\theta)$, $n \neq 0$ yields

$$I \exp(in\theta) = \{ \exp(in\theta) \} / (in),$$

or equivalently,

$$If(\theta) = \int_0^\theta d\theta' f(\theta') + \int_0^{2\pi} \theta' d\theta' f(\theta')/(2\pi) \quad (A17)$$

and it is readily verified that If is periodic of period 2π , $\langle If \rangle = 0$, and $(d/d\theta)(If) = f(\theta)$.

In (A16) we now replace $\tilde{p}_3^2 + \tilde{q}_3^2$ by 2μ and consider the formula to define a Hamiltonian $H(\tilde{p}_1, \tilde{q}_1, \tilde{p}_2, \tilde{q}_2)$. If we introduce the generator

$$F = \bar{p}_1 \tilde{q}_1 + \bar{p}_2 \tilde{q}_2 + IA_\lambda (\epsilon^{3/2} \bar{p}_1 + \epsilon \tilde{q}_2, \epsilon \bar{p}_2, \tilde{q}_1 + \sqrt{\epsilon} \bar{p}_2) / \sqrt{\epsilon}, \quad (A18)$$

where the variables $(\bar{p}_1, \bar{q}_1)$, $(\bar{p}_2, \bar{q}_2)$ are not to be confused with those defined by (A1) and (A2), then we find

$$\begin{aligned} \tilde{p}_1 &= \bar{p}_1 + A_\lambda (\epsilon^{3/2} \bar{p}_1 + \epsilon \tilde{q}_2, \epsilon \bar{p}_2, \tilde{q}_1 + \sqrt{\epsilon} \bar{p}_2) / \sqrt{\epsilon} \\ \tilde{q}_1 &= \tilde{q}_1 + \epsilon I A_{\lambda, \psi} (\epsilon^{3/2} \bar{p}_1 + \epsilon \tilde{q}_2, \epsilon \bar{p}_2, \tilde{q}_2 + \sqrt{\epsilon} \bar{p}_2) \\ \tilde{p}_2 &= \bar{p}_2 + \sqrt{\epsilon} I A_{\lambda, \psi} (\epsilon^{3/2} \bar{p}_1 + \epsilon \tilde{q}_2, \epsilon \bar{p}_2, \tilde{q}_1 + \sqrt{\epsilon} \bar{p}_2) \\ \tilde{q}_2 &= \tilde{q}_2 + A_\lambda (\epsilon^{3/2} \bar{p}_1 + \epsilon \tilde{q}_2, \epsilon \bar{p}_2, \tilde{q}_1 + \sqrt{\epsilon} \bar{p}_2) \\ &\quad + \sqrt{\epsilon} I A_{\lambda, \lambda} (\epsilon^{3/2} \bar{p}_1 + \epsilon \tilde{q}_2, \epsilon \bar{p}_2, \tilde{q}_1 + \sqrt{\epsilon} \bar{p}_2) \text{ and} \\ H &= \mu \{ B + \epsilon R E / (2B) \} + (\sqrt{\epsilon} \tilde{q}_2)^2 (R - \epsilon R^2 E / B) / 2 \\ &\quad + \epsilon \mu B^2 ((A_\psi / B)_{, \lambda} - (A_\lambda / B)_{, \psi}) - \epsilon R A_\lambda A_{\psi, \theta} (\sqrt{\epsilon} \tilde{q}_2) \\ &\quad + \epsilon (\sqrt{\epsilon} \tilde{q}_2)^2 (R_{, \psi} A_\lambda - R_\lambda A_\psi) / 2 + O(\epsilon^{3/2}), \end{aligned} \quad (A19)$$

and the flux coordinate is not changed in an essential way:

$$\epsilon^{3/2} \bar{p}_1 + \epsilon \tilde{q}_2 = \epsilon^{3/2} \tilde{p}_1 + \epsilon \tilde{q}_2 + O(\epsilon^{3/2}),$$

where the error is a time independent point function not representing a particle drift. We could easily carry this computation to higher order if needed or desired.

One final transformation, akin to the treatment of (A13) simplifies the structure of the Hamiltonian (A19) considerably. We let

$f(\psi, \lambda, \theta)$ and $g(\psi, \lambda, \theta)$ be two arbitrary continuous functions, periodic in θ and with zero average and we introduce the generator

$$F = \hat{p}_1 \bar{q}_1 + \hat{p}_2 \bar{q}_2 + I F + \sqrt{\epsilon} \bar{q}_2 I g, \quad (A20)$$

where $I f$ and $I g$ are evaluated at ψ, λ, θ with $\psi = \epsilon^{3/2} \hat{p}_1 + \epsilon \bar{q}_2$, $\lambda = \epsilon \hat{p}_2$ and $\theta = \bar{q}_1 + \sqrt{\epsilon} \hat{p}_2$. If we set

$$g = (R_{,\psi} A_\lambda - R_{,\lambda} A_\psi) / (2R)$$

$$f = A_\lambda A_{\psi,\theta} - \langle A_\lambda A_{\psi,\theta} \rangle$$

then

$$\begin{aligned} & H(\epsilon^{3/2} \hat{p}_1 + \epsilon \hat{q}_1, \epsilon \hat{p}_2, \sqrt{\epsilon} \hat{q}_2, \hat{q}_1 + \sqrt{\epsilon} \hat{p}_2) \\ &= \mu(B + \epsilon R E / (2B)) + (\sqrt{\epsilon} \hat{q}_2)^2 (R - \epsilon R^2 E / B) / 2 \\ &\quad - \epsilon R (\sqrt{\epsilon} \hat{q}_2) \langle A_\lambda A_{\psi,\theta} \rangle + \epsilon \mu B^2 ((A_\psi / B)_{,\lambda} - (A_\lambda / B)_{,\psi}) + O(\epsilon^{3/2}) \end{aligned} \quad (A21)$$

functions of space are evaluated at (ψ, λ, θ) and

$$\begin{aligned} \psi &= \epsilon^{3/2} \hat{p}_1 + \epsilon \hat{q}_2 \\ \lambda &= \epsilon \hat{p}_2 \\ \theta &= \hat{q}_1 + \sqrt{\epsilon} \hat{p}_2. \end{aligned} \quad (A22)$$

The scaling for our variables is $\epsilon^{3/2} \hat{p}_1$, $\epsilon \hat{p}_2$ and $\sqrt{\epsilon} \hat{q}_2$ are all $O(1)$ and we do not need to scale $\hat{q}_1$. Further, the flux coordinate of a particle is

$$\psi = \epsilon^{3/2} \hat{p}_1 + \epsilon \hat{q}_2 + O(\epsilon^{3/2}), \quad (A23)$$

where the error is independent of time and does not represent a particle drift, but rather change of coordinates.

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Footnotes

- (1) The research monograph by T. G. Northrop, *The Adiabatic Motion of Charged Particles* (Wiley, New York 1963), contains essentially all the older results and has a thorough bibliography.
- (2) See, for instance J. A. Rome and Y-K. M. Peng, *Nucl. Fusion* 19, 1193 (1979) and A. H. Boozer, *Phys. Fluids* 23, 904 (1980).
- (3) C. S. Gardner, *Phys. Rev.* 115, 791 (1959).
- (4) T. G. Northrop and J. A. Rome, *Phys. Fluids* 21, 384 (1978).
- (5) Private communication. A. H. Boozer, see also his work cited in note (2).
- (6) E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Fourth edition (Dover Publications, New York, 1944), Section 141, p. 313.
- (7) E. T. Whittaker, *Ibid.*, Section 131, p. 300.

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**N.Y.U. Courant Institute of
Mathematical Sciences**

251 Mercer St.

New York, N. Y. 10012

